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CHARACTERIZING CONSISTENCY
BY MONOMIALS AND BY PRODUCT DISPERSIONS

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ABSTRACT. This paper derives two characterizations of the Kreps-Wilson concept of consistent beliefs. In the first, beliefs are shown to be consistent iff they can be constructed from the elements of monomial vectors which converge to the strategies. In the second, beliefs are shown to be consistent iff they can be induced by a product dispersion whose marginal dispersions induce the strategies. The first characterization is simpler than the definition in Kreps and Wilson (1982), and the second seems more fundamental in the sense that it is built on an underlying theory of relative probability.

1. INTRODUCTION

This paper will draw extensively from Kreps and Wilson (1982) (henceforth KW). KW's definition of sequential equilibrium includes the definition of consistency, and that definition says that beliefs and strategies are consistent with one another iff they are the limit of a sequence of strictly positive beliefs and strategies which are consistent with one another via Bayes Rule. This paper provides two characterizations of this important definition.

The first characterization (Theorem 2.1) says that beliefs and strategies are consistent iff there are monomial vectors such that (a) the strategy at each information set is the limit of the monomial vector at that information set, and (b) the belief at each information set is found by calculating the product of the monomials along the paths leading to each of the nodes in the information set. This simplifies the KW definition because each action is assigned a monomial (i.e., a

coefficient and an exponent) rather than a sequence (which is infinite-dimensional). Streufert (2006b) uses the theorem to repair a nontrivial fallacy in KW's proofs, and Subsection 2.2 discusses how it extends a result in Perea y Monsuwe, Jansen, and Peters (1997).

The second characterization (Theorem 3.1) is very disaggregated. It concerns the Cartesian product of the action sets across the many information sets, together with an additional dimension for the initial nodes. The theorem says that beliefs and strategies are consistent iff there is a product dispersion over this Cartesian product such that (a1) the exogenous distribution across the initial nodes is induced by the corresponding marginal of the product, (a2) the strategy at each information set is induced by the corresponding marginal of the product, and (b) the belief at each information set is induced by the restriction of the product to those elements of the Cartesian product that correspond to the information set. In my eyes, this construction is fundamental: it is unaffected by arbitrary choices of sequences or monomials, and is instead based upon a theory of relative probability (Streufert (2006a)). Subsection 3.5 discusses how the construction stems from Kohlberg and Reny (1997) and how it relates to Fudenberg and Tirole (1991).

These two characterizations are derived simultaneously (Theorem 4.1). This is surprising because the two characterizations appear to depart from the KW definition in opposite directions: the first uses monomial vectors to reduce complexity while the second uses large product dispersions that seem to increase complexity. Yet, the two are directly linked via an underlying result in Streufert (2006a) which shows that a dispersion can be represented by a product of monomial vectors iff it is a product dispersion.

2. Characterization by Monomials

2.1. Basic Definitions

Section 2 uses KW notation. This subsection recapitulates certain KW definitions and introduces an example which will be used throughout the paper.

This paragraph and Figure 2.1 define a game form $[T, \prec, A, \alpha, H, \rho]$. The set T of nodes contains the set $X = \{o, oL, oLd, oR\}$ of decision nodes, which in turn contains the set $W = \{o\}$ of initial nodes. The set W is given the trivial distribution $\rho = (\rho(o)) = (1)$, and the set X is partitioned by the information sets $h \in H = \{\{o\}, \{oL\}, \{oLd, oR\}\}$. Let $H(x)$ denote the information set h which contains x. Finally, let
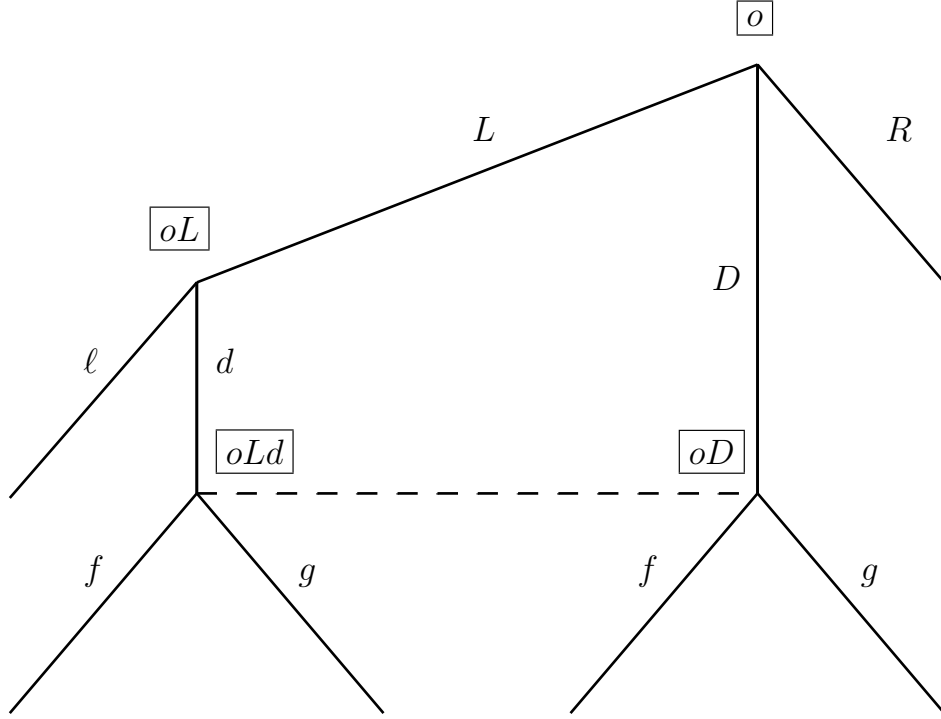
FIGURE 2.1

$A = \{L, D, R, \ell, d, f, g\}$ be the set of actions a, let $A(h)$ be the set of actions available from information set h, and let $\alpha(x)$ be the last action taken to reach a non-initial node x.

A strategy profile is a function $\pi{:}A{\to}[0,1]$ such that $(\forall h)$ $\Sigma_{a \in A(h)}\pi(a)$ $= 1$ (we assume perfect recall and consider only behavioural strategies). A belief system is a function $\mu{:}X{\to}[0,1]$ such that $(\forall h)$ $\Sigma_{x \in h}\mu(x) = 1$. An assessment is a pair (π, μ). Let Ψ^0 consist of those strictly positive assessments which satisfy

$$(1) \qquad (\forall x)\ \mu(x) = \frac{\rho \circ p_{\ell(x)}(x) \cdot \Pi_{k=0}^{\ell(x)-1} \pi \circ \alpha \circ p_k(x)}{\Sigma_{x' \in H(x)}\ \rho \circ p_{\ell(x')}(x') \cdot \Pi_{k=0}^{\ell(x')-1} \pi \circ \alpha \circ p_k(x')}\ ,$$

where $p_k(x)$ is the kth predecessor of node x, and $\ell(x)$ is the number of its predecessors. An assessment (μ, π) is said to be *consistent* if it is the limit of a sequence $\langle(\mu^n, \pi^n)\rangle_n$ in Ψ^0. For instance, in the example, the (π, μ) defined by the second lines of

		L	D	R	ℓ	d	f	g
(2)	$\pi^n(a)$	$\frac{n^{-1}}{1+3n^{-1}}$	$\frac{2n^{-1}}{1+3n^{-1}}$	$\frac{1}{1+3n^{-1}}$	$\frac{1}{1+6n^{-2}}$	$\frac{6n^{-2}}{1+6n^{-2}}$	$\frac{1}{2}$	$\frac{1}{2}$
	$\pi(a)$	0	0	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$

	o	oL	oLd	oD
$\mu^n(x)$	1	1	$\frac{6n^{-3}}{2n^{-1}+6n^{-3}}$	$\frac{2n^{-1}}{2n^{-1}+6n^{-3}}$
$\mu(x)$	1	1	0	1

(3)   and

is consistent because the second line in each table is the limit of its first line, and because the (π^n, μ^n) defined in the first line of both tables is within Ψ^0 for any value of n.

2.2. Theorem

Theorem 2.1 characterizes consistency by means of two functions defined over the set A of actions. The function e assigns an integer "exponent" to each action, and the function c assigns a positive real "coefficient" to each action. This is simpler than the KW definition because two functions of A are simpler than a sequence of functions of A.

Theorem 2.1. *In any game form* $[T, \prec, A, \alpha, \rho, H]$, *an assessment* (μ, π) *is consistent iff there exists* $c{:}A{\to}(0, \infty)$ *and* $e{:}A{\to}\mathbb{Z}$ *such that*

$$(\forall a) \ \pi(a) = \lim_{n\to\infty} c(a)n^{e(a)} \ , \ and$$

$$(\forall x) \ \mu(x) = \lim_{n\to\infty} \frac{\rho\circ p_\ell(x)\cdot\Pi_{k=0}^{\ell(x)-1}c\circ\alpha\circ p_k(x)n^{e\circ\alpha\circ p_k(x)}}{\Sigma_{x'\in H(x)} \ \rho\circ\ell(x')\cdot\Pi_{k=0}^{\ell(x')-1}c\circ\alpha\circ p_k(x')n^{e\circ\alpha\circ p_k(x')}} \ .$$

Proof. Theorem 4.1($b{\Leftrightarrow}c$) by means of the notational modifications around (7) and (8). $\square$

Theorem 2.1 is equivalent to a reformulation of Lemmas A1 and A2 on KW pages 887 and 888. Nonetheless, Theorem 2.1 is valuable because it repairs a nontrivial fallacy in KW's proof of these lemmas (details in Streufert (2006b)), and because these lemmas in turn provide the logical basis for KW's three theorems about the geometry of the set of sequential equilibrium assessments, the finiteness of the set of sequential equilibrium outcomes, and the perfection of strict sequential equilibria. The paper by Perea y Monsuwe, Jansen, and Peters (1997) appears to recognize neither its close relation to the KW lemmas nor the fallacy in the KW proof, and its Theorem 3.1 is weaker than the KW lemmas to the extent that it derives the analog of real but not necessarily integer exponents (details in Appendix A).

The functions c and e can be together regarded as a single function which assigns a monomial $c(a)n^{e(a)}$ to each action a. For instance, the first line in the following table defines a monomial at each action in the
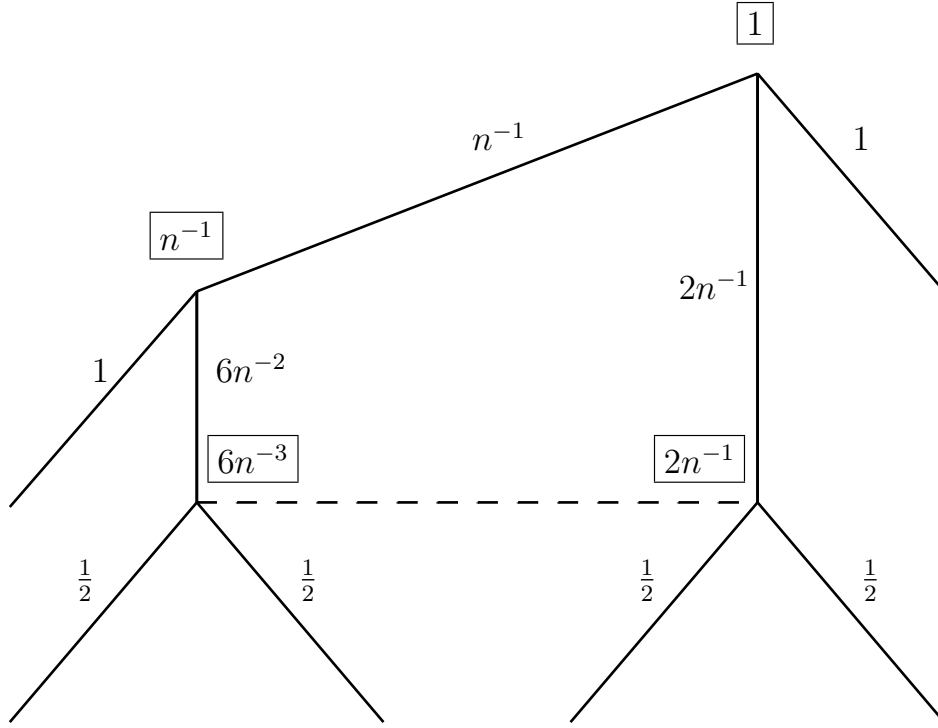
FIGURE 2.2

example

$$(4) \quad \begin{array}{c|ccc|cc|cc} a & L & D & R & \ell & d & f & g \\ \hline c(a)n^{e(a)} & n^{-1} & 2n^{-1} & 1 & 1 & 6n^{-2} & .5 & .5 \\ \pi(a) & 0 & 0 & 1 & 1 & 0 & .5 & .5 \end{array} \quad .$$

The second line is then the strategy derived via the Theorem 2.1's first equation.

The theorem's second equation asks one to calculate a product at each node. Fortunately, this product is just the product of the monomials along the path leading to the node. For instance, in Figure 2.2, the unboxed monomial at each action is taken from the first line of (4) and the boxed monomial at each node is the product of the unboxed monomials above it. These boxed monomials appear in the first line of

$$(5) \quad \begin{array}{c|c|c|cc} x & o & oL & oLd & oD \\ \hline \rho \circ p_{\ell(x)}(x) \cdot \Pi_{k=0}^{\ell(x)-1} c \circ \alpha \circ p_k(x) n^{e \circ \alpha \circ p_k(x)} & 1 & n^{-1} & 6n^{-3} & 2n^{-1} \\ \mu(x) & 1 & 1 & 0 & 1 \end{array} \quad .$$

The second line is then the belief derived via the theorem's second equation.

By Theorem 2.1, the assessment (π, μ) defined in (4) and (5) is consistent. This is rather uninteresting because (4) and (5) are very similar to (2) and (3). In fact, it is always the case that the monomials defined by c and e determine a special kind of sequence $\langle \pi^n \rangle_n$ by means of

$$(\forall h)(\forall a \in A(h)) \ \pi^n(a) = \frac{c(a)n^{e(a)}}{\Sigma_{a' \in A(h)}c(a')n^{e(a')}} \ .$$

However, the converse provided by Theorem 2.1 is valuable. It shows that *any* consistent assessment can be supported with this special kind of sequence.

The following corollary is equivalent to Theorem 2.1. In both the theorem and the corollary, the first equation uses the exponents to determine the support of the strategy at each h and then uses the coefficients to determine the probabilities over that support. Similarly, the second equation uses the exponents to determine the support of the belief at each h and then uses the coefficients to determine the probabilities over that support. The corollary's formulation makes these observations more apparent.

COROLLARY 2.2. *In any game form* $[T, \prec, A, \alpha, \rho, H]$, *an assessment* (μ, π) *is consistent iff there exists* $c:A \to (0, \infty)$ *and* $e:A \to \mathbb{Z}$ *such that* $(\forall a) \ e(a) \leq 0$,

$$(\forall a) \ \pi(a) = \begin{pmatrix} c(a) & \text{if } e(a) = 0 \\ 0 & \text{if } e(a) < 0 \end{pmatrix} , \ and$$

$$(\forall x) \ \mu(x) = \begin{pmatrix} \dfrac{\rho \circ p_{\ell(x)}(x) \cdot \Pi_{k=0}^{\ell(x)-1} c \circ \alpha \circ p_k(x)}{\Sigma_{x' \in H^e(x)} \rho \circ p_{\ell(x')}(x') \cdot \Pi_{k=0}^{\ell(x)-1} c \circ \alpha \circ p_k(x')} & \text{if } x \in H^e(x) \\ 0 & \text{if } x \notin H^e(x) \end{pmatrix}$$

where $H^e(x) = argmax\{ \Sigma_{k=0}^{\ell(x')-1} e \circ \alpha \circ p_k(x') \mid x' \in H(x) \}$.

Proof. The first equation of Theorem 2.1 is equivalent to the nonpositivity of e and the first equation of Corollary 2.2. The second equations of the two results are equivalent. □

3. CHARACTERIZATION BY PRODUCT DISPERSIONS

3.1. MODIFIED BASIC NOTATION

The remainder of this paper will use the modified notation that is introduced in Table 3.1. The big picture is that Theorems 3.1 and 4.1 are best understood in terms of vectors of the form $[x_i]$ and matrices of the form $[x_{ij}]$.

	KW and Section 2	Sections 3 and 4
information set	h	h
a node in h		x_h
the set of decision nodes	X	
a decision node	x	
a belief at h	$\mu\|_h$	$[\mu_{x_h}]$
a belief system	μ	$([\mu_{x_h}])_h$
the set of actions at h	$A(h)$	A_h
an action at h		a_h
the set of actions	A	
an action	a	
a strategy at h	$\pi\|_{A(h)}$	$[\pi_{a_h}]$
a strategy profile	π	$([\pi_{a_h}])_h$

TABLE 3.1

In addition, we will frequently need to derive a distribution from a vector of functions. Specifically, let Z be a finite set, let $[\nu_z]$ be a distribution over Z, and let $[f_z(n)]$ be a vector over Z of functions $f_z(n)$ of n. Then say that $[\nu_z]$ is *induced* by $[f_z(n)]$ if

$$(6) \qquad (\forall z)\ \nu_z = \lim_{n\to\infty} \frac{f_z(n)}{\Sigma_{z'}\, f_{z'}(n)}\ .$$

To exercise this new notation and terminology, note that the assessment $([\pi_{a_h}])_h, ([\mu_{x_h}])_h$ is consistent (as defined near (1)) iff there exists a profile $(\langle[\pi_{a_h}^n]\rangle_n)_h$ of full-support distribution sequences $\langle[\pi_{a_h}^n]\rangle_n$ such that

$$(7) \qquad (\forall h)\ [\pi_{a_h}] = \lim_{n\to\infty}[\pi_{a_h}^n] \text{ and}$$

$$(\forall h)\ [\mu_{x_h}] \text{ is induced by } [\rho_{p_\ell(x_h)}\cdot\Pi_{k=0}^{\ell(x)-1}\pi_{\alpha\circ p_k(x_h)}^n]\ ,$$

and that by Theorem 1, this is equivalent to the existence of a profile $([c_{a_h} n^{e_{a_h}}])_h$ of monomial vectors $[c_{a_h} n^{e_{a_h}}]$ such that

$$(8) \qquad (\forall h)\ [\pi_{a_h}] = \lim_{n\to\infty}[c_{a_h} n^{e_{a_h}}] \text{ and}$$

$$(\forall h)\ [\mu_{x_h}] \text{ is induced by } [\rho_{p_\ell(x_h)}\cdot\Pi_{k=0}^{\ell(x)-1} c_{\alpha\circ p_k(x_h)} n^{e_{\alpha\circ p_k(x_h)}}]\ .$$

3.2. DISPERSIONS

Streufert (2005, Section 2) introduces dispersions. Here is a brief synopsis. Consider any finite set Z. A *table* over Z is a $[q_{z/z'}] \in [0,\infty]^{Z^2}$

which lists a relative probability $q_{z/z'} \in [0, \infty]$ for every pair of elements z and z' from Z. A *dispersion* over Z is a table $[q_{z/z'}]$ that satisfies $(\forall z) \, q_{z/z} = 1$ and

$$(\forall z, z', z'') \, q_{z/z''} \in \odot(q_{z/z'}, q_{z'/z''}) \, ,$$

in which $\odot$ is a set-valued function assigning subsets of $[0, \infty]$ to pairs $(u, v) \in [0, \infty]^2$ according to the rule

$$\odot(u, v) = \left(\begin{array}{ll} [0, \infty] & \text{if } (u, v) \text{ equals } (0, \infty) \text{ or } (\infty, 0) \\ \{uv\} & \text{otherwise} \end{array} \right) \, .$$

A dispersion $[q_{z/z'}]$ *induces* the distribution $[\nu_z]$ satisfying

$$(9) \qquad\qquad (\forall z) \, \nu_z = \frac{q_{z/z^*}}{\Sigma_{z' \in Z} \, q_{z'/z^*}} \, ,$$

for some $z^* \in Z$ satisfying $(\forall z' \in Z) \, q_{z'/z^*} < \infty$. In other words a dispersion induces the distribution that is derived by normalizing any "row" of the dispersion that contains only finite relative probabilities. Streufert (2005, Remark 2.1) shows that every dispersion induces exactly one distribution.

3.3. PRODUCTS

Streufert (2005, Section 3) and Streufert (2006a, Section 2) introduce products over a nonempty finite collection $(Z_i)_{i=1}^{\ell}$ of nonempty finite sets Z_i. A *product* over $(Z_i)_{i=1}^{\ell}$ is table over the Cartesian product $Z = \Pi_{i=1}^{\ell} Z_i$ which belongs to the set $\Delta(Z_i)_{i=1}^{\ell}$ defined by

$$(10) \qquad\qquad \Delta(Z_i)_{i=1}^{\ell} = \{ \, [q_{z/z'}] \in [0, \infty]^{Z^2} \, |$$

$$(\forall m)(\forall \sigma)(\forall (z^j)_{j=0}^{m}) \, 1 \in \odot(q_{z^{\sigma,j}/z^j})_{j=0}^{m} \, \} \, ,$$

in which m is a nonnegative integer, σ is a vector $(\sigma_1, \sigma_2, ... \sigma_\ell)$ of permutations of $\{0, 1, ... m\}$, each z^j is a vector in the Cartesian product $Z = \Pi_{i=1}^{\ell} Z_i$, each $z^{\sigma,j}$ is the vector in Z defined by

$$z^{\sigma,j} = (z_1^{\sigma_1(j)}, z_2^{\sigma_2(j)}, ... z_\ell^{\sigma_\ell(j)}) \, ,$$

and $\odot$ is the function which assigns a subset of $[0, \infty]$ to every $(u^j)_{j=0}^{m}$ in $[0, \infty]^{1+m}$ according to the rule

$$\odot(u^j)_{j=0}^{m} = \left(\begin{array}{ll} [0, \infty] & \text{if } (\exists j)u^j = 0 \text{ and } (\exists j)u^j = \infty \\ \{\Pi_{j=0}^{m} u^j\} & \text{otherwise} \end{array} \right)$$

($\{1\}$ is assigned to the empty vector). Streufert (2006a, Remark 3.2) observes that every product is a dispersion (and hence "product" and "product dispersion" are synonymous.)

The *marginals* of a product $[q_{z/z'}]$ over $(Z_i)_{i=1}^{\ell}$ are the ℓ dispersions $([q_{z_i/z_i'}])_{i=1}^{\ell}$ which satisfy

$$(\forall z, z') \ q_{z/z'} \in \odot(q_{z_i/z_i'})_{i=1}^{\ell} \ .$$

Streufert (2006a, Remark 4.1) shows that every product has a unique vector of marginals, and that for any dimension i, the marginal $[q_{z_i/z_i'}]$ satisfies

(11) $$[q_{z_i/z_i'}] = [q_{z_i z_{-i}^{\star}/z_i' z_{-i}^{\star}}]$$

for any $z_{-i}^{\star} \in \Pi_{j \neq i} Z_j$. Note that marginals are defined to be dispersions (and hence "marginal" and "marginal dispersion" are synonymous).

The more leisurely discussion of producthood in Streufert (2005, Section 3) makes two general observations about producthood which might bear repeating here. First, a product is defined to be a table over a Cartesian product in which cancellations can occur in its different dimensions independently. In this regard, product dispersions are like product distributions. Second, many different products can share the same vector of marginals. In other words, a vector of marginal dispersions is ambiguous. In this regard, product dispersions are different from product distributions (in my own experience, this is difficult to remember).

3.4. Informal Introduction to Theorem

Streufert (2005, Sections 4 and 5) introduces this paper's second characterization of consistency in a simpler setting in which the only nontrivial information set follows after two simultaneous moves. Accordingly, all but the bravest souls might like to get comfortable with statement (b) in that paper's Theorem 5.1 before venturing further.

In the present setting, consider the example and contemplate the Cartesian product

(12) $$W \times \Pi_h A_h = \{o\} \times \{L, D, R\} \times \{\ell, d\} \times \{f, g\} =$$

$$\{ \ oL\ell g, \ oLdg, \ oD\ell g, \ oDdg, \ oR\ell g, \ oRdg,$$
$$oL\ell f, \ oLdf, \ oD\ell f, \ oDdf, \ oR\ell f, \ oRdf \ \}$$

The following theorem is concerned with all the relative probabilities among the 12 elements of this set. Hence it is concerned with a 144-dimensional dispersion. One of these creatures is lurking in Figure 3.2. Notice that its rows and columns are labelled with the elements of the Cartesian product (12). For instance, Figure 3.2 says that $q_{oD\ell f/oL\ell g} = 2$.

	oLℓf	oLℓg	oLdf	oLdg	oDℓf	oDℓg	oDdf	oDdg	oRℓf	oRℓg	oRdf	oRdg
$oRdg$	0	0	0	0	0	0	0	0	∞	∞	1	1
$oRdf$	0	0	0	0	0	0	0	0	∞	∞	1	1
$oR\ell g$	0	0	0	0	0	0	0	0	1	1	0	0
$oR\ell f$	0	0	0	0	0	0	0	0	1	1	0	0
$oDdg$	∞	∞	.5	.5	∞	∞	1	1	∞	∞	∞	∞
$oDdf$	∞	∞	.5	.5	∞	∞	1	1	∞	∞	∞	∞
$oD\ell g$	.5	.5	0	0	1	1	0	0	∞	∞	∞	∞
$oD\ell f$	.5	.5	0	0	1	1	0	0	∞	∞	∞	∞
$oLdg$	∞	∞	1	1	∞	∞	2	2	∞	∞	∞	∞
$oLdf$	∞	∞	1	1	∞	∞	2	2	∞	∞	∞	∞
$oL\ell g$	1	1	0	0	2	2	0	0	∞	∞	∞	∞
$oL\ell f$	1	1	0	0	2	2	0	0	∞	∞	∞	∞
	$oL\ell f$	$oL\ell g$	$oLdf$	$oLdg$	$oD\ell f$	$oD\ell g$	$oDdf$	$oDdg$	$oR\ell f$	$oR\ell g$	$oRdf$	$oRdg$

TABLE 3.2

The "third" player's strategy can be derived from this dispersion. In particular, the "third" player chooses at the information set $h = \{oLd, oD\}$ among the options $A_h = \{f, g\}$. By (11), the marginal of the dispersion with respect to $A_h = \{f, g\}$ is

$$[q_{oL\ell a_h/oL\ell a'_h}] = \begin{bmatrix} q_{oL\ell f/oL\ell g} & q_{oL\ell g/oL\ell g} \\ q_{oL\ell f/oL\ell f} & q_{oL\ell g/oL\ell f} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and this marginal induces the strategy $(\pi_f, \pi_g) = (.5, .5)$. Note that this 2×2 marginal appears in the bottom-left corner of Figure 3.2.

There was, by the way, nothing special about using $oL\ell$ to find the marginal with respect to $\{f, g\}$. We might have used oLd, $oD\ell$, oDd, $oR\ell$, or oRd. The five corresponding 2×2 tables can be found along the main diagonal in Figure 3.2. As expected from (11), they are all equal.

The "second" player's strategy can also be derived from this dispersion. In particular, the "second" player chooses at the information set $h = \{oL\}$ among the options $A_h = \{\ell, d\}$. By (11), the marginal with respect to $\{\ell, d\}$ is

$$[q_{oLa_h f/oLa'_h f}] = \begin{bmatrix} q_{oL\ell f/oLdf} & q_{oLdf/oLdf} \\ q_{oL\ell f/oL\ell f} & q_{oLdf/oL\ell f} \end{bmatrix} = \begin{bmatrix} ∞ & 1 \\ 1 & 0 \end{bmatrix},$$

and this induces the strategy $(\pi_\ell, \pi_d) = (1, 0)$. Note that this 2×2 marginal appears in the four corners of the 3×3 table in the bottom-left corner of Figure 3.2 (there are five equal 2×2 tables corresponding to oL_g, oD_f, oD_g, oR_f, and oR_g).

Finally, the "first" player's strategy can also be derived from the dispersion. In particular, the "first" player chooses at the information set $h = \{o\}$ among the options $A_h = \{L, D, R\}$. By (11), the marginal with respect to $\{L, D, R\}$ is

$$[q_{oa_h\ell f/oa_h'\ell f}] = \begin{bmatrix} q_{oL\ell f/oR\ell f} & q_{oD\ell f/oR\ell f} & q_{oR\ell f/oR\ell f} \\ q_{oL\ell f/oD\ell f} & q_{oD\ell f/oD\ell f} & q_{oR\ell f/oD\ell f} \\ q_{oL\ell f/oL\ell f} & q_{oD\ell f/oL\ell f} & q_{oR\ell f/oL\ell f} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ .5 & 1 & \infty \\ 1 & 2 & \infty \end{bmatrix} ,$$

which induces the strategy $(\pi_L, \pi_D, \pi_R) = (0, 0, 1)$. To find this 3×3 marginal within Figure 3.2, note the nine rectangles partitioning the table and take the bottom left element from each of them (there are three equal 3×3 tables corresponding to $o\_\ell g$, o_df, and o_dg).

It remains to find the belief at the third player's information set $h = \{oLd, oD\}$. We need two sets. The first is

$$\begin{aligned}
S_{oLd} &= \{ \, w(a_\eta)_\eta \mid w = p_{\ell(oLd)}(oLd) \text{ and} \\
&\qquad (\forall \ell \in \{0, 1, \dots \ell(oLd)-1\} \; a_{H \circ p_{\ell+1}(oLd)} = \alpha \circ p_\ell(oLd) \, \} \\
&= \{ \, w(a_\eta)_\eta \mid w = p_2(oLd), \\
&\qquad a_{H \circ p_2(oLd)} = \alpha \circ p_1(oLd), \; a_{H \circ p_1(oLd)} = \alpha \circ p_0(oLd) \, \} \\
&= \{ \, w(a_\eta)_\eta \mid w = o, \; a_{H(o)} = \alpha(oL), \; a_{H(oL)} = \alpha(oLd) \, \} \\
&= \{ \, w(a_\eta)_\eta \mid w = o, \; a_{\{o\}} = L, \; a_{\{oL\}} = d \, \} \\
&= \{oLdf, oLdg\} \, ,
\end{aligned}$$

which is the subset of the Cartesian product $\{o\}\times\{L, D, R\}\times\{\ell, d\}\times\{f, g\}$ which is "compatible" with the node oLd. Similarly, the subset of the Cartesian product which is "compatible" with oD is

$$\begin{aligned}
S_{oD} &= \{ \, w(a_\eta)_\eta \mid w = p_{\ell(oD)}(oD) \text{ and} \\
&\qquad (\forall \ell \in \{0, 1, \dots \ell(oD)-1\} \; a_{H \circ p_{\ell+1}(oD)} = \alpha \circ p_\ell(oD) \, \} \\
&= \{ \, w(a_\eta)_\eta \mid w = p_1(oD), \; a_{H \circ p_1(oD)} = \alpha \circ p_0(oD) \, \} \\
&= \{ \, w(a_\eta)_\eta \mid w = o, \; a_{H(o)} = \alpha(oD) \, \} \\
&= \{ \, w(a_\eta)_\eta \mid w = o, \; a_{\{o\}} = D \, \} \\
&= \{oD\ell f, oD\ell g, oDdf, oDdg\} \, .
\end{aligned}$$

The union of these two sets,

$$S_{oLD} \cup S_{oD} = \{oLdf, oLdg, oD\ell f, oD\ell g, oDdf, oDdg\} \ ,$$

is comprised of those elements of the Cartesian product that might have something to do with the belief at the information set $h = \{oLd, oD\}$.

Accordingly, consider the restriction of Figure 3.2 to $(S_{oLd} \cup S_{oD})^2$. This restriction is the $6{\times}6$ boxed table within Figure 3.2. This restriction induces the following distribution over $S_{oLd} \cup S_{oD}$

		S_{oLd}		S_{oD}			
(13)	$w(a_\eta)_\eta$	$oLdf$	$oLdg$	$oD\ell f$	$oD\ell g$	$oDdf$	$oDdg$
	$\nu_{w(a_\eta)_\eta \mid S_{oLd} \cup S_{oD}}$	0	0	.5	.5	0	0

and this distribution implies that the belief over $h = \{oLd, oD\}$ is

$$(\ \nu_{S_{oLd} \mid S_{oLd} \cup S_{oD}}, \ \nu_{S_{oD} \mid S_{oLd} \cup S_{oD}} \ ) = (0, 1) \ .$$

To summarize, the product dispersion in Figure 3.2 determines all the strategies and all the beliefs. In particular, its marginals determine the strategies and its restrictions determine the beliefs. Producthood plays an essential role in these calculations by assuring that the marginals are well-defined via (11).

Streufert (2005, Section 5.2) summarizes an example which illustrates that a strategy does not uniquely determine a marginal, and further, that a profile of marginals does not uniquely determine a product. These two sources of ambiguity are the reasons that one strategy profile can be consistent with many belief systems.

3.5. Formal Theorem

Consider any information set h. At any x_h, define

$$(14) \qquad S_{x_h} = \{ \ w(a_\eta)_\eta \in W {\times} \Pi_\eta A_\eta \mid w = p_{\ell(x_h)}(x_h) \text{ and}$$
$$(\forall \ell \in \{0, 1, \dots \ell(x_h){-}1\}) \ a_{H \circ p_{\ell+1}(x_h)} = \alpha \circ p_\ell(x_h) \ \} \ .$$

Then let $[q_{w(a_\eta)_\eta / w'(a'_\eta)_\eta}]|_{\cup_{x'_h} S_{x'_h}}$ denote the restriction of the dispersion $[q_{w(a_\eta)_\eta / w'(a'_\eta)_\eta}]$ to $\cup_{x'_h} S_{x'_h}$. This restriction induces a conditional distribution over $\cup_{x'_h} S_{x'_h}$ which we will denote $[\nu_{w(a_\eta)_\eta \mid \cup_{x'_h} S_{x'_h}}]$. This conditional distribution then determines the probability of each S_{x_h} relative to $\cup_{x'_h} S_{x'_h}$ by

$$(15) \qquad (\forall x_h) \ \nu_{S_{x_h} \mid \cup_{x'_h} S_{x'_h}} = \Sigma_{w(a_\eta)_\eta \in S_{x_h}} \ \nu_{w(a_\eta)_\eta \mid \cup_{x'_h} S_{x'_h}}$$

(the assumption of perfect recall guarantees that $(S_{x_h})_{x_h}$ partitions $\cup_{x'_h} S_{x'_h}$).

THEOREM 3.1. *In any game form* $[T, \prec, A, \alpha, \rho, H]$*, an assessment* $([\pi_{a_h}])_h, ([\mu_{x_h}])_h$ *is consistent iff there exists a product* $[q_{w(a_h)_h / w'(a'_h)_h}]$ *over* $(W, (A_h)_h)$ *such that*

> $(\rho, ([\pi_{a_h}])_h)$ *is induced by the marginals of* $[q_{w(a_h)_h / w'(a'_h)_h}]$ *and*
>
> $(\forall h)\ [\mu_{x_h}]$ *is the* $[\nu_{S_{x_h} | \cup_{x'_h} S_{x'_h}}]$ *induced by* $[q_{w(a_h)_h / w'(a'_h)_h}]|_{\cup_{x'_h} S_{x'_h}}$.

Proof. Theorem 4.1($c \Leftrightarrow d$).   □

The inspiration for this characterization can be traced to Kohlberg and Reny (1997). In particular, a reformulation of its Theorem 2.10 is equivalent to Streufert (2006a, Remark 6.3) (further discussion appears there).

The above characterization is also related to an idea pursued by Fudenberg and Tirole (1991, Section 6). That paper considers the equivalent of a dispersion on the set of terminal nodes. This differs from the above characterization in two ways. First, it is more aggregated: the set of terminal nodes is relatively small and expands to the Cartesian product $W \times \Pi_h A_h$ only in simultaneous-move games. Second, and much more importantly, dispersionhood is far weaker than producthood, and producthood with its many cancellation laws corresponds to the KW definition of consistency (this is the underlying issue identified by Kohlberg and Reny (1997) in their note 17).

I would suggest that product dispersions provide a comparatively fundamental way of understanding consistency. Unlike the KW definition of consistency, it is uncluttered with arbitrary choices of sequences. And, unlike Theorem 1's characterization, it is uncluttered with arbitrary choices of coefficients and exponents.

4. GENERAL THEOREM

4.1. A THIRD CHARACTERIZATION

Statement (a) in Theorem 4.1 provides a third characterization of consistency. It is close to (b), which was discussed in Section 2 as the first of this paper's two characterizations. In fact, (b) is derived from a convenient "normalization" of (a). Although relatively unimportant, (a) does have some independent value as a sufficient condition for consistency when one would rather not be bothered with the normalization inherent in (b).

Statement (a) is also a focal point of the Theorem 4.1's underlying logic. At first glance, it is useful to notice that the theorem's four equivalent statements are arranged so that all three "downhill" implications are much easier than getting from (d) all the way back to (a). This difficult step is achieved via Streufert (2006a), whose Theorem 5.1 proves that any product dispersion (such as $[q_{w(a_h)_h/w'(a'_h)_h}]$) can be represented by some product of monomial vectors (such as $[c_w n^{e_w} \Pi_h c_{a_h} n^{e_{a_h}}]$). Once (a) has been attained, it is comparatively easy to find a "normalization" of the monomials which yields the relatively simple expressions of (b). Then, another sort of "normalization" converts these expressions into the strategy sequences defining consistency in (c).

Since the reader is probably familiar with the definition of consistency in (c), the proof starts at (c) and immediately proceeds to derive (d)'s underlying product dispersion from the definition of consistency. It then makes the leap to (a), drops to (b), and drops back to (c).

THEOREM 4.1. *Let* $[T, \prec, A, \alpha, \rho, H]$ *be a game form. Then the following four statements are equivalent for any assessment* $([\pi_{a_h}], [\mu_{x_h}])_h$.
(a) There exists a $([c_{a_h} n^{e_{a_h}}])_h$ *such that*

$$(\forall h)\ [\pi_{a_h}]\ \text{is induced by}\ [c_{a_h} n^{e_{a_h}}]\ \text{and}$$
$$(\forall h)\ [\mu_{x_h}]\ \text{is induced by}\ [\Sigma_{w(a_\eta)_\eta \in S_{x_h}} \rho_w \Pi_\eta c_{a_\eta} n^{e_{a_\eta}}]\ .$$

(b) There exists a $([c_{a_h} n^{e_{a_h}}])_h$ *such that*

$$(\forall h)\ [\pi_{a_h}] = lim_{n\to\infty} [c_{a_h} n^{e_{a_h}}]\ \text{and}$$
$$(\forall h)\ [\mu_{x_h}]\ \text{is induced by}\ [\rho_{p_{\ell(x_h)}(x_h)} \cdot \Pi_{k=0}^{\ell(x_h)-1} c_{\alpha \circ p_k(x_h)} n^{e_{\alpha \circ p_k(x_h)}}]\ .$$

(c) (Consistency in KW) There exists a $(\langle [\pi_{a_h}^n] \rangle_n)_h$ *such that*

$$(\forall h)\ [\pi_{a_h}] = lim_{n\to\infty} [\pi_{a_h}^n]\ \text{and}$$
$$(\forall h)\ [\mu_{x_h}]\ \text{is induced by}\ [\rho_{p_{\ell(x_h)}(x_h)} \cdot \Pi_{k=0}^{\ell(x_h)-1} \pi_{\alpha \circ p_k(x_h)}^n]\ .$$

(d) There exists a product $[q_{w(a_h)_h/w'(a'_h)_h}]$ *over* $(W, (A_h)_h)$ *such that*

$$(\rho, ([\pi_{a_h}])_h)\ \text{is induced by the marginals of}\ [q_{w(a_h)_h/w'(a'_h)_h}]\ \text{and}$$
$$(\forall h)\ [\mu_{x_h}]\ \text{is the}\ [\nu_{S_{x_h}|\cup_{x'_h} S_{x'_h}}]\ \text{induced by}\ [q_{w(a_h)_h/w'(a'_h)_h}]|_{(\cup_{x'_h} S_{x'_h})^2}\ .$$

The word "induce" is littered throughout the theorem. In (a), (b), and (c), a distribution is "induced" by a vector of functions of n according to definition (6). In (d), a distribution is "induced" by a finite row in a dispersion according to definitions (9) and (15).

4.2.   PROOF OF THEOREM 4.1(C⇒D)

The substantive matter here is that the sequences defining consistency in (c) imply the cancellation laws defining producthood in (d). This matter is addressed in the first two paragraphs of Proof 4.4 below.

The remainder of the proof is concerned with translating from the game-tree notation of (c) to the product notation of (d). This translation is facilitated by a definition and two lemmas. A table $[q_{z/z'}]$ is said to be *approximated* by a sequence $\langle[\beta_z^n]\rangle_n$ of positive vectors if

(16) $$(\forall z, z') \; q_{z/z'} = \lim_{n\to\infty}\beta_z^n/\beta_{z'}^n \; .$$

Because this concept concerns ratios, the positive vectors $[\beta_z^n]$ need not be normalized as full-support distributions. It is well understood that the existence of an approximation is equivalent to dispersionhood (see Streufert (2005, Note 7) for details).

LEMMA 4.2. *Suppose that $\langle[\beta_z^n]\rangle_n$ approximates $[q_{z/z'}]$. Then $\langle[\beta_z^n]\rangle_n$ induces exactly one distribution, $[q_{z/z'}]$ induces exactly one distribution, and these two distributions are identical.*

Proof. Streufert (2005, Remark 2.1) yields that $[q_{z/z'}]$ induces exactly one distribution. This is one of the lemma's three conclusions. Let $[\nu_z^Q]$ denote this unique distribution, and note from the definition (9) of inducement that there exists a z^* such that

(17a) $$(\forall z) \; \nu_z^Q = q_{z/z^*}/\Sigma_{z'}q_{z'/z^*} \text{ and}$$

(17b) $$(\forall z') \; q_{z'/z^*} < \infty \; .$$

Then

(18) $$(\forall z) \; \nu_z^Q =_1 \frac{q_{z/z^*}}{\Sigma_{z'} \; q_{z'/z^*}}$$

$$=_2 \frac{\lim_{n\to\infty}(\beta_z^n/\beta_{z^*}^n)}{\Sigma_{z'}\lim_{n\to\infty}(\beta_{z'}^n/\beta_{z^*}^n)} =_3 \frac{\lim_{n\to\infty}(\beta_z^n/\beta_{z^*}^n)}{\lim_{n\to\infty}\Sigma_{z'}(\beta_{z'}^n/\beta_{z^*}^n)}$$

$$=_4 \lim_{n\to\infty}\frac{(\beta_z^n/\beta_{z^*}^n)}{\Sigma_{z'}(\beta_{z'}^n/\beta_{z^*}^n)} =_5 \lim_{n\to\infty}\frac{\beta_z^n}{\Sigma_{z'}\beta_{z'}^n} \; ,$$

where $=_1$ holds by (17a), $=_2$ holds by the lemma's assumption of approximation and the definition (16), $=_3$ holds by the algebra of limits and (17), $=_4$ holds by the algebra of limits and the fact that the denominator is at least $\beta_{z^*}^n/\beta_{z^*}^n = 1$, and $=_5$ holds by algebra.

Equation (18) yields two conclusions. First, it shows that $[\nu_z^Q]$ is induced by $\langle[\beta_z^n]\rangle_n$, and thus $\langle[\beta_z^n]\rangle_n$ induces at least one distribution. Second, if $[\nu_z^B]$ is *any* distribution induced by $\langle[\beta_z^n]\rangle_n$, then $[\nu_z^B]$

must equal the right-hand side of (18), and hence, (18) yields that $[\nu_z^B]$ equals $[\nu_z^Q]$, which is the unique distribution induced by $[q_{z/z'}]$. Hence, $\langle[\beta_z^n]\rangle_n$ induces exactly one distribution, and that distribution equals the unique distribution induced by $[q_{z/z'}]$. $\square$

LEMMA 4.3. *Suppose a product over $(Z_i)_{i=1}^{\ell}$ is approximated by a sequence of the form $\langle[\Pi_{i=1}^{\ell}\beta_{z_i}^n]\rangle_n$. Then, for any i, its marginal with respect to Z_i is appoximated by $\langle[\beta_{z_i}^n]\rangle_n$.*

Proof. Let $[q_{z/z'}]$ be the product approximated by $\langle[\Pi_{i=1}^{\ell}\beta_{z_i}^n]\rangle_n$. Fix any $z^\star$, and consider any dimension i. First, (11) yields that the marginal with respect to z_i equals $[q_{z_i z_{-i}^\star/z_i' z_{-i}^\star}]$. Second, since $[q_{z_i z_{-i}^\star/z_i' z_{-i}^\star}]$ is a restriction of $[q_{z/z'}]$, and since all of $[q_{z/z'}]$ is approximated by $\langle[\Pi_{j=1}^{\ell}\beta_{z_j}^n]\rangle_n$, we know $[q_{z_i z_{-i}^\star/z_i' z_{-i}^\star}]$ is approximated by $\langle[\beta_{z_i}^n](\Pi_{j\neq i}\beta_{z_j^\star}^n)\rangle_n$ (remember that $z_{-i}^\star$ is fixed). These two sentences together yield that the marginal with respect to z_i is approximated by $\langle[\beta_{z_i}^n](\Pi_{j\neq i}\beta_{z_j^\star}^n)\rangle_n$. Thus, since the definition (16) of approximation depends only on ratios, the marginal with respect to z_i is also approximated by $\langle[\beta_{z_i}^n]\rangle_n$. $\square$

PROOF 4.4 (for Theorem 4.1($c\Rightarrow d$)). Assume (c). For any n, define the table $[q_{w(a_h)_h/w'(a_h')_h}^n]$ over $W\times\Pi_h A_h$ by

$$(19)\qquad (\forall w(a_h)_h, w'(a_h')_h)\quad q_{w(a_h)_h/w'(a_h')_h}^n = \frac{\rho_w\Pi_h\pi_{a_h}^n}{\rho_{w'}\Pi_h\pi_{a_h'}^n}\ ,$$

Notice that every such table is a product over $(W,(A_h)_h)$ because the table's definition ensures that all of the cancellation laws in (10) are satisfied by means of ordinary algebra (every relative probability is positive and finite).

Now consider the sequence $\langle[q_{w(a_h)_h/w'(a_h')_h}^n]\rangle_n$ of such tables. This is a sequence in $(|W|\cdot\Pi_h|A_h|)^2$ dimensions and there is no reason to believe that it will converge. However, the sequence lies in the space of products over $(W,(A_h)_h)$ (by the last sentence of the last paragraph) and the space of all products over $(W,(A_h)_h)$ is compact (by Streufert (2006a, Theorem 6.1)). Accordingly, a subsequence converges to a product over $(W,(A_h)_h)$. Let $\langle[q_{w(a_h)_h/w'(a_h')_h}^m]\rangle_m$ denote the subsequence, and let $[q_{w(a_h)_h/w'(a_h')_h}]$ denote its limit. Hence, by (19), we have that $\langle[\rho_w\Pi_h\pi_{a_h}^m]\rangle_m$ approximates the product $[q_{w(a_h)_h/w'(a_h')_h}]$.

First Half. By the previous sentence and Lemma 4.3, the marginal dispersion of $[q_{w(a_h)_h/w'(a_h')_h}]$ with respect to w is approximated by the

constant sequence $\langle[\rho_w]\rangle_m$, and the marginal dispersion of this product with respect to each a_h is approximated by $\langle[\pi^m_{a_h}]\rangle_m$.

Note (rather easily) that $[\rho_w]$ is induced by the dispersion that is approximated by the constant sequence $\langle[\rho_w]\rangle_m$. Since this dispersion is the marginal with respect to w (by the previous paragraph), we have that $[\rho_w]$ is induced by the marginal with respect to w. Now consider any h. Since $\langle[\pi^m_{a_h}]\rangle_m$ approximates the marginal with respect to a_h (by the previous paragraph) and since $[\pi_{a_h}]$ is induced by $\langle[\pi^m_{a_h}]\rangle_m$ (by the first half of (c)), we have that $[\pi_{a_h}]$ is induced by the marginal with respect to a_h (by Lemma 4.2). Thus the first half of (d) holds.

Second Half. Now consider any h. Define

$$(\forall x_h) \; N_{x_h} = \{ \; \eta \mid \text{not } (\exists a_\eta)(\exists k \in \{0, 1, \ldots \ell(x_h)-1\}) \; a_\eta = \alpha \circ p_k(x_h) \; \}$$

(thus N_{x_h} contains those information sets which are *not* reached on the way to x_h). Note that

$$(20) \qquad (\forall x_h)(\forall m) \; \Sigma\{\Pi_{\eta \in N_{x_h}} \pi^m_{a_\eta} | (a_\eta)_{\eta \in N_{x_h}}\} = 1$$

since $[\Pi_{\eta \in N_{x_h}} \pi^m_{a_\eta}]$ is an ordinary product distribution over $\Pi_{\eta \in N_{x_h}} A_\eta$ simply because every $[\pi^m_{a_\eta}]$ is a distribution over A_η. This leads to

$$
\begin{aligned}
(\forall x_h)(\forall m) \quad & \rho_{p_{\ell(x_h)}(x_h)} \cdot \Pi_{k=0}^{\ell(x_h)-1} \pi^m_{\alpha \circ p_k(x_h)} \\
(21) \qquad =_1 \; & \rho_{p_{\ell(x_h)}(x_h)} \cdot \Pi_{k=0}^{\ell(x_h)-1} \pi^m_{\alpha \circ p_k(x_h)} \cdot \Sigma\{\Pi_{\eta \in N_{x_h}} \pi^m_{a_\eta} | (a_\eta)_{\eta \in N_{x_h}}\} \\
=_2 \; & \Sigma\{ \rho_{p_{\ell(x_h)}(x_h)} \cdot \Pi_{k=0}^{\ell(x_h)-1} \pi^m_{\alpha \circ p_k(x_h)} \cdot \Pi_{\eta \in N_{x_h}} \pi^m_{a_\eta} | (a_\eta)_{\eta \in N_{x_h}}\} \\
=_3 \; & \Sigma_{w(a_\eta)_\eta \in S_{x_h}} p_w \Pi_\eta \pi^m_{a_\eta} \; ,
\end{aligned}
$$

where $=_1$ holds by (20), $=_2$ by algebra, and $=_3$ by the definition (14) of S_{x_h}.

Since $\langle[\rho_w \Pi_h \pi^m_{x_h}]\rangle_m$ approximates $[q_{w(a_\eta)_\eta / w'(a'_\eta)_\eta}]$ (by the second paragraph of the proof), the restriction of the sequence to $\cup_{x'_h} S_{x'_h}$ approximates the restriction of the dispersion to $(\cup_{x'_h} S_{x'_h})^2$. Hence, by Lemma 4.2, we may let the left-hand side of (22) denote the unique distribution induced by the restriction of the sequence to $\cup_{x'_h} S_{x'_h}$, let right-hand side of (22) denote the unique distribution induced by the restriction of the dispersion to $(\cup_{x'_h} S_{x'_h})^2$, and record for future reference that

$$(22) \qquad\qquad [\nu^{(c)}_{w(a_\eta)_\eta | \cup_{x'_h} S_{x'_h}}] = [\nu^{(d)}_{w(a_\eta)_\eta | \cup_{x'_h} S_{x'_h}}] \; .$$

This paragraph uses (21) and (22) to derive the second half of (d). In particular,

$$(\forall x_h) \quad \mu_{x_h}$$

$$=_1 \lim_{m\to\infty} \frac{\rho_{p_{\ell(x_h)}(x_h)} \Pi_{k=0}^{\ell(x_h)-1} \pi_{\alpha\circ p_k(x_h)}^m}{\Sigma_{x_h'} \rho_{p_{\ell(x_h')}(x_h')} \Pi_{k=0}^{\ell(x_h')-1} \pi_{\alpha\circ p_k(x_h')}^m}$$

$$=_2 \lim_{m\to\infty} \frac{\Sigma_{w(a_\eta)_\eta \in S_{x_h}} p_w \Pi_\eta \pi_{a_\eta}^m}{\Sigma_{x_h'} \Sigma_{w(a_\eta)_\eta \in S_{x_h}} p_w \Pi_\eta \pi_{a_\eta}^m}$$

$$=_3 \lim_{m\to\infty} \Sigma_{w(a_\eta)_\eta \in S_{x_h}} \frac{p_w \Pi_\eta \pi_{a_\eta}^m}{\Sigma_{w(a_\eta)_\eta \in \cup_{x_h'} S_{x_h'}} p_w \Pi_\eta \pi_{a_\eta}^m}$$

$$=_4 \Sigma_{w(a_\eta)_\eta \in S_{x_h}} \lim_{m\to\infty} \frac{p_w \Pi_\eta \pi_{a_\eta}^m}{\Sigma_{w(a_\eta)_\eta \in \cup_{x_h'} S_{x_h'}} p_w \Pi_\eta \pi_{a_\eta}^m}$$

$$=_5 \Sigma_{w(a_\eta)_\eta \in S_{x_h}} \nu_{w(a_\eta)_\eta | \cup_{x_h'} S_{x_h'}}^{(c)}$$

$$=_6 \Sigma_{w(a_\eta)_\eta \in S_{x_h}} \nu_{w(a_\eta)_\eta | \cup_{x_h'} S_{x_h'}}^{(d)} \quad,$$

where $=_1$ holds by the second half of (c), $=_2$ holds by (21), $=_3$ holds by algebra, $=_4$ holds by the algebra of limits and the fact that every term being summed is less than one, $=_5$ holds by the definition of $[\nu_{w(a_\eta)_\eta | \cup_{x_h'} S_{x_h'}}^{(c)}]$ above (22), and $=_6$ holds by (22). By the definition of $[\nu_{w(a_\eta)_\eta | \cup_{x_h'} S_{x_h'}}^{(d)}]$ above (22) and by the definition of ν at (15), the entire equality is the second half of (d). $\qquad\square$

4.3. PROOF OF THEOREM 4.1(D⇒A)

This is the critical part of the proof. Its essential ingredient is Streufert (2006a, Theorem 5.1), which shows that every product dispersion can be represented by a product of monomial vectors. This result will be applied to $[q_{w(a_h)_h/w'(a_h')_h}]$ in order to obtain $[c_w n^{e_w} \cdot \Pi_h c_{a_h} n^{e_{a_h}}]$.

In order to employ this theorem, we require a definition and two lemmas. As in Streufert (2006a, equation (15)), a monomial vector $[c_z n^{e_z}]$ *represents* the table $[q_{z/z'}]$ defined by

$$(23) \qquad (\forall z, z') \; q_{z/z'} = \lim_{n\to\infty} \frac{c_z n^{e_z}}{c_{z'} n^{e_{z'}}} = \begin{pmatrix} \infty & \text{if } e_z > e_{z'} \\ c_z/c_{z'} & \text{if } e_z = e_{z'} \\ 0 & \text{if } e_z < e_{z'} \end{pmatrix}$$

(the second equality is an obvious fact). It is well understood that the existence of a representation is equivalent to dispersionhood (see

Streufert (2005, Note 4) for details). The following lemma notes that representation is invariant to monomial multiplication.

LEMMA 4.5. *For any monomial ξn^ε and any monomial vector $[c_z n^{e_z}]$, the table represented by $[c_z n^{e_z}]$ equals the table represented by $\xi n^\varepsilon [c_z n^{e_z}]$.*

Proof. If $[q_{z/z'}]$ is represented by $[c_z n^{e_z}]$ and $[q^\star_{z/z'}]$ is represented by $\xi n^\varepsilon [c_z n^{e_z}]$, then (23) implies

$$(\forall z, z') \ q_{z/z'} = \lim_{n\to\infty} \frac{c_z n^{e_z}}{c_z n^{e_z}} = \lim_{n\to\infty} \frac{\xi n^\varepsilon c_z n^{e_z}}{\xi n^\varepsilon c_z n^{e_z}} = q^\star_{z/z'} \ .$$

$\square$

LEMMA 4.6. *Suppose $[c_z n^{e_z}]$ represents $[q_{z/z'}]$. Then $[c_z n^{e_z}]$ induces exactly one distribution, $[q_{z/z'}]$ induces exactly one distribution, and these two distributions are identical.*

Proof. Any $[c_z n^{e_z}]$ induces exactly one distribution because the limit in the definition (6) of induction must exist when the functions of n are monomials. Any $[q_{z/z'}]$ induces exactly one distribution by Streufert (2005, Remark 2.1). Thus, if $[c_z n^{e_z}]$ represents $[q_{z/z'}]$, the induced distributions are identical by Streufert (2005, Lemma 5.2). $\square$

PROOF 4.7 (for Theorem 4.1(d⇒a)). Assume (d). By Theorem 5.1(b⇒a) of Streufert (2006a), the product $[q_{w(a_h)_h/w'(a'_h)_h}]$ is represented by some $[c_w n^{e_w} \cdot \Pi_h c_{a_h} n^{e_{a_h}}]$. Since representation is invariant to monomial multiplication by Lemma 4.5, we may multiply this original $[c_w n^{e_w} \cdot \Pi_h c_{a_h} n^{e_{a_h}}]$ by the monomial

$$(\Sigma_{w\in\text{argmax}\{e_\omega|\omega\}} c_w)^{-1} n^{-\max\{e_\omega|\omega\}}$$

to arrive at a "normalized" $[c_w n^{e_w} \cdot \Pi_h c_{a_h} n^{e_{a_h}}]$ which both represents $[q_{w(a_h)_h/w'(a'_h)_h}]$ and satisfies

(24) $\max\{e_w\} = 0$ and $\Sigma_w\{c_w|e_w{=}0\} = 1$.

Further, by the second sentence of Streufert (2006a, Theorem 5.1), the marginals of the product $[q_{w(a_h)_h/w'(a'_h)_h}]$ are represented by $[c_w n^{e_w}]$ and $([c_{a_h} n^{e_h}])_h$.

First Half. Consider any h. By the last sentence of the previous paragraph, $[c_{a_h} n^{e_h}]$ represents the marginal with respect to a_h. Thus, by Lemma 4.6, the distribution induced by $[c_{a_h} n^{e_h}]$ equals the distribution induced by the marginal with respect to a_h. The latter is $[\pi_{a_h}]$ by the first half of (d). Hence the former is $[\pi_{a_h}]$ as well. In other words, $[\pi_{a_h}]$ is the distribution induced by $[c_{a_h} n^{e_h}]$. This establishes the first half of (a).

Second Half. By the last sentence of the next-to-last paragraph, $[c_w n^{ew}]$ represents the marginal with respect to w. Thus, by Lemma 4.6, the distribution induced by $[c_w n^{ew}]$ equals the distribution induced by the marginal with respect to w. The latter is $[\rho_w]$ by the first half of (d). Hence, the former is $[\rho_w]$ as well. In other words, $[\rho_w]$ is the distribution induced by $[c_w n^{ew}]$. By the normalization (24) and the assumption that $[\rho_w]$ has full support, it must be that

$$(25) \qquad\qquad [\rho_w] = [c_w n^{ew}] \ .$$

Now consider any h. Since the monomial vector $[c_w n^{ew} \cdot \Pi_h c_{a_h} n^{e a_h}]$ represents the product $[q_{w(a_h)_h / w'(a'_h)_h}]$ (by the first paragraph), the restriction of the monomial vector to $\cup_{x'_h} S_{x'_h}$ represents the restriction of the product to $(\cup_{x'_h} S_{x'_h})^2$. Thus, by Lemma 4.6, we may let the left-hand side of (26) denote the unique distribution induced by the restriction of the monomial vector, let the right-hand side of (26) denote the unique distribution induced by the restriction of the product, and record for future reference that

$$(26) \qquad\qquad [\nu^{(a)}_{w(a_\eta)_\eta | \cup_{x'_h} S_{x'_h}}] = [\nu^{(d)}_{w(a_\eta)_\eta | \cup_{x'_h} S_{x'_h}}] \ .$$

This paragraph uses (25) and (26) to derive the second half of (a). In particular,

$$(\forall x_h) \quad \mu_{x_h}$$
$$=_1 \ \Sigma_{w(a_h)_h \in S_{x_h}} \ \nu^{(d)}_{w(a_h)_h | \cup_{x'_h} S_{x'_h}}$$
$$=_2 \ \Sigma_{w(a_h)_h \in S_{x_h}} \ \nu^{(a)}_{w(a_h)_h | \cup_{x'_h} S_{x'_h}}$$
$$=_3 \ \Sigma_{w(a_h)_h \in S_{x_h}} \lim_{n \to \infty} \frac{c_w n^{ew} \cdot \Pi_\eta c_{a_\eta} n^{e a_\eta}}{\Sigma_{w'(a'_h)_h \in \cup_{x'_h} S_{x'_h}} c_{w'} n^{ew'} \cdot \Pi_\eta c_{a'_\eta} n^{e a'_\eta}}$$
$$=_4 \ \lim_{n \to \infty} \frac{\Sigma_{w(a_h)_h \in S_{x_h}} c_w n^{ew} \cdot \Pi_\eta c_{a_\eta} n^{e a_\eta}}{\Sigma_{w'(a'_h)_h \in \cup_{x'_h} S_{x'_h}} c_{w'} n^{ew'} \cdot \Pi_\eta c_{a'_\eta} n^{e a'_\eta}}$$
$$=_5 \ \lim_{n \to \infty} \frac{\Sigma_{w(a_h)_h \in S_{x_h}} \rho_w \cdot \Pi_\eta c_{a_\eta} n^{e a_\eta}}{\Sigma_{w'(a'_h)_h \in \cup_{x'_h} S_{x'_h}} \rho_{w'} \cdot \Pi_\eta c_{a'_\eta} n^{e a'_\eta}} \ ,$$

where $=_1$ holds by the second sentence of (d), the definition of ν at (15) and the definition of $[\nu^{(d)}_{w(a_\eta)_\eta | \cup_{x'_h} S_{x'_h}}]$ above (26), $=_2$ holds by (26), $=_3$ holds by definition of $[\nu^{(a)}_{w(a_\eta)_\eta | \cup_{x'_h} S_{x'_h}}]$ above (26), $=_4$ holds by the algebra of limits and the fact that all the limits on the left-hand side are less

than 1, and $=_5$ holds by (25). The entire equality is the second half of (a). □

4.4. PROOF OF THEOREM 4.1(A⇒B)

Essentially, this step simplifies (a)'s algebra to arrive at (b). The trick is to use Lemma 4.8 to find a convenient "normalization" of the monomial vectors $([c_{a_h}n^{e_{a_h}}])_h$. Intuitively, this trick becomes clear in the first two paragraphs of Proof 4.9. Unfortunately, the details are taxing because the expressions determining beliefs have zero-limit denominators at every zero-probability information set.

LEMMA 4.8. *For any monomial ξn^{ε} and any monomial vector $[c_z n^{e_z}]$, the distribution induced by $[c_z n^{e_z}]$ is the same as the distribution induced by $\xi n^{\varepsilon}[c_z n^{e_z}]$.*

Proof. Suppose $[\nu_z]$ is induced by $[c_z n^{e_z}]$ and $[\nu_z^{\star}]$ is induced by $\xi n^{\varepsilon}[c_z n^{e_z}]$. Then by the definition (6) of inducement,

$$(\forall z)\ \nu_z = \lim_{n\to\infty}\frac{c_z n^{e_z}}{\Sigma_{z'}c_{z'}n^{e_{z'}}} = \lim_{n\to\infty}\frac{\xi n^{\varepsilon}c_z n^{e_z}}{\Sigma_{z'}\xi n^{\varepsilon}c_{z'}n^{e_{z'}}} = \nu_z^{\star}\ .$$

□

PROOF 4.9 (for Theorem 4.1(a⇒b)). Assume (a). Because induction is invariant to monomial multiplication by Lemma 4.8, we may multiply each original $[c_{a_h}n^{e_{a_h}}]$ by the monomial

$$(\Sigma\{\ c_{a_h}\mid a_h \in \text{argmax}\{e_{a_h'}|a_h'\}\ \})^{-1}n^{-\max\{e_{a_h'}|a_h'\}}$$

in order to arrive at a "normalized" $([c_{a_h}n^{e_{a_h}}])_h$ which satisfies both halves of (a) as well as

(27a) $$(\forall h)\ \max\{e_{a_h}|a_h\} = 0 \text{ and}$$
(27b) $$(\forall h)\ \Sigma\{c_{a_h}|e_{a_h}=0\} = 1\ .$$

First Half. The first half of (b) holds because

(28) $$(\forall a_h)\quad \pi_{a_h}$$
$$=_1 \lim_{n\to\infty} c_{a_h}n^{e_{a_h}}/\Sigma_{a_h'}c_{a_h'}n^{e_{a_h'}}$$
$$=_2 \begin{pmatrix} c_{a_h}/\Sigma\{c_{a_h'}|e_{a_h'}=0\} & \text{if } e_{a_h}=0 \\ 0 & \text{if } e_{a_h}<0 \end{pmatrix}$$
$$=_3 \begin{pmatrix} c_{a_h} & \text{if } e_{a_h}=0 \\ 0 & \text{if } e_{a_h}<0 \end{pmatrix}$$
$$=_4 \lim_{n\to\infty} c_{a_h}n^{e_{a_h}}\ ,$$

where $=_1$ holds by the first half of (a), $=_2$ holds by (27a), $=_3$ holds by (27b), and $=_4$ holds by (27a).

Second Half. Fix h. The first task is to set up a way of dealing with zero-limit denominators: define

$$e_h = \max\{ \ \Sigma_\eta e_{a_\eta} \ | \ w(a_\eta)_\eta \in \cup_{x'_h} S_{x'_h} \ \} \ ,$$

and note that

(29a) $\qquad (\forall x_h) \ \lim_{n\to\infty} \ n^{-e_h} \cdot \Sigma_{w(a_\eta)_\eta \in S_{x_h}} \rho_w \Pi_\eta c_{a_\eta} n^{e a_\eta} \ \in [0,\infty)$ and

(29b) $\qquad (\exists x_h) \ \lim_{n\to\infty} \ n^{-e_h} \cdot \Sigma_{w(a_\eta)_\eta \in S_{x_h}} \rho_w \Pi_\eta c_{a_\eta} n^{e a_\eta} \ \in (0,\infty) \ .$

Then, for any x_h, define the set

(30) $\qquad N_{x_h} = \{ \ \eta \ | \ \text{not } (\exists a_\eta)(\exists k \in \{0,1,...\ell(x)-1\}) \ a_\eta = \alpha \circ p_k(x) \ \}$

(thus N_{x_h} consists of the information sets through which one does *not* pass on the way to node x_h). We can make two observations. First,

(31) $\qquad \lim_{n\to\infty} \ \Sigma\{ \ \Pi_{\eta \in N_{x_h}} c_{a_\eta} n^{e a_\eta} \ | \ (a_\eta)_{\eta \in N_{x_h}} \ \}$

$\qquad =_1 \ \Sigma\{ \ \Pi_{\eta \in N_{x_h}} \lim_{n\to\infty} c_{a_\eta} n^{e a_\eta} \ | \ (a_\eta)_{\eta \in N_{x_h}} \ \}$

$\qquad =_2 \ \Sigma\{ \ \Pi_{\eta \in N_{x_h}} \pi_{x_h} \ | \ (a_\eta)_{\eta \in N_{x_h}} \ \}$

$\qquad =_3 \ 1 \ ,$

where $=_1$ holds by (28) and the algebra of limits, $=_2$ holds by (28), and $=_3$ holds because $\Pi_{\eta \in N_{x_h}} \pi_{x_h}$ is an ordinary product distribution over the Cartesian product $\Pi_{\eta \in N_{x_h}} A_\eta$. Second,

(32) $\quad (\forall x_h) \quad \Sigma_{w(a_\eta)_\eta \in S_{x_h}} \rho_w \Pi_\eta c_{a_\eta} n^{e a_\eta}$

$\qquad =_1 \Sigma\{ \ \rho_w \Pi_\eta c_{a_\eta} n^{e a_\eta} \ | \ w = p_{\ell(x_h)}(x_h) \text{ and}$

$\qquad\qquad (\forall k \in \{0,1,...\ell(x_h)-1\}) \ a_{H \circ p_{k+1}(x_h)} = \alpha \circ p_k(x_h) \ \}$

$\qquad =_2 \Sigma\{ \ \rho_{p_{\ell(x_h)}(x_h)} \cdot \Pi_{k=0}^{\ell(x_h)-1} c_{\alpha \circ p_k(x_h)} n^{e \alpha \circ p_k(x_h)} \cdot \Pi_{\eta \in N_{x_h}} c_{a_\eta} n^{e a_\eta}$

$\qquad\qquad | \ (a_\eta)_{\eta \in N_{x_h}} \ \}$

$\qquad =_3 \rho_{p_{\ell(x_h)}(x_h)} \cdot \Pi_{k=0}^{\ell(x_h)-1} c_{\alpha \circ p_k(x_h)} n^{e \alpha \circ p_k(x_h)} \cdot$

$\qquad\qquad \Sigma\{ \ \Pi_{\eta \in N_{x_h}} c_{a_\eta} n^{e a_\eta} \ | \ (a_\eta)_{\eta \in N_{x_h}} \ \} \ .$

where $=_1$ holds by the definition (14) of S_{x_h}, $=_2$ holds by the definition (30) of N_{x_h}, and $=_3$ holds by algebra.

This paragraph uses (31) and (32) to derive the second half of (b). It takes two steps. First,

(33) $\qquad (\forall x_h) \quad \lim_{n\to\infty} \ n^{-e_h} \cdot \Sigma_{w(a_\eta)_\eta \in S_{x_h}} \rho_w \Pi_\eta c_{a_\eta} n^{e a_\eta}$

$$=_1 \ \lim_{n\to\infty} \ n^{-e_h}\cdot\rho_{p_{\ell(x)}(x)}\cdot\Pi_{k=0}^{\ell(x)-1}c_{\alpha\circ p_k(x)}n^{e_{\alpha\circ p_k(x)}}.$$

$$\Sigma\{\ \Pi_{\eta\in N_{x_h}}c_{a_\eta}n^{e_{a_\eta}}\ |\ (a_\eta)_{\eta\in N_{x_h}}\ \}$$

$$=_2 \ \lim_{n\to\infty} \ n^{-e_h}\cdot\rho_{p_{\ell(x)}(x)}\cdot\Pi_{k=0}^{\ell(x)-1}c_{\alpha\circ p_k(x)}n^{e_{\alpha\circ p_k(x)}}.$$

$$\lim_{n\to\infty} \Sigma\{\ \Pi_{\eta\in N_{x_h}}c_{a_\eta}n^{e_{a_\eta}}\ |\ (a_\eta)_{\eta\in N_{x_h}}\ \}$$

$$=_3 \ \lim_{n\to\infty} \ n^{-e_h}\cdot\rho_{p_{\ell(x)}(x)}\cdot\Pi_{k=0}^{\ell(x)-1}c_{\alpha\circ p_k(x)}n^{e_{\alpha\circ p_k(x)}}\ ,$$

where $=_1$ holds by (32), $=_2$ holds (29a), (31), and the algebra of limits, and $=_3$ holds by (31). Second,

$$(\forall x_h)\quad \mu_{x_h}$$

$$=_1 \ \lim_{n\to\infty}\frac{\Sigma_{w(a_\eta)_\eta\in S_{x_h}}\rho_w\Pi_\eta c_{a_\eta}n^{e_{a_\eta}}}{\Sigma_{w(a_\eta)_\eta\in\cup_{x_h'}S_{x_h'}}\rho_w\Pi_\eta c_{a_\eta}n^{e_{a_\eta}}}$$

$$=_2 \ \lim_{n\to\infty}\frac{n^{-e_h}\cdot\Sigma_{w(a_\eta)_\eta\in S_{x_h}}\rho_w\Pi_\eta c_{a_\eta}n^{e_{a_\eta}}}{\Sigma_{x_h'}\ n^{-e_h}\cdot\Sigma_{w(a_\eta)_\eta\in S_{x_h'}}\rho_w\Pi_\eta c_{a_\eta}n^{e_{a_\eta}}}$$

$$=_3 \ \frac{\lim_{n\to\infty}\ n^{-e_h}\cdot\Sigma_{w(a_\eta)_\eta\in S_{x_h}}\rho_w\Pi_\eta c_{a_\eta}n^{e_{a_\eta}}}{\Sigma_{x_h'}\lim_{n\to\infty}\ n^{-e_h}\cdot\Sigma_{w(a_\eta)_\eta\in S_{x_h'}}\rho_w\Pi_\eta c_{a_\eta}n^{e_{a_\eta}}}$$

$$=_4 \ \frac{\lim_{n\to\infty}\ n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}c_{\alpha\circ p_k(x_h)}n^{e_{\alpha\circ p_k(x_h)}}}{\Sigma_{x_h'}\lim_{n\to\infty}\ n^{-e_h}\cdot\rho_{p_{\ell(x_h')}(x_h')}\cdot\Pi_{k=0}^{\ell(x_h')-1}c_{\alpha\circ p_k(x_h')}n^{e_{\alpha\circ p_k(x_h')}}}$$

$$=_5 \ \lim_{n\to\infty}\frac{n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}c_{\alpha\circ p_k(x_h)}n^{e_{\alpha\circ p_k(x_h)}}}{\Sigma_{x_h'}\ n^{-e_h}\cdot\rho_{p_{\ell(x_h')}(x_h')}\cdot\Pi_{k=0}^{\ell(x_h')-1}c_{\alpha\circ p_k(x_h')}n^{e_{\alpha\circ p_k(x_h')}}}$$

$$=_6 \ \lim_{n\to\infty}\frac{\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}c_{\alpha\circ p_k(x_h)}n^{e_{\alpha\circ p_k(x_h)}}}{\Sigma_{x_h'}\ \rho_{p_{\ell(x_h')}(x_h')}\cdot\Pi_{k=0}^{\ell(x_h')-1}c_{\alpha\circ p_k(x_h')}n^{e_{\alpha\circ p_k(x_h')}}}$$

where $=_1$ is the first half of (a), $=_2$ follows from algebra, $=_3$ follows from (29) and the algebra of limits, $=_4$ follows from (33), $=_5$ follows from (29), (33) and the algebra of limits, and $=_6$ follows by algebra. The entire equality is the second half of (b). $\square$

4.5. Proof of Theorem 4.1(b⇒c)

This step transforms each monomial vector $[c_{x_h}n^{e_{x_h}}]$ into a strategy sequence $\langle[\pi_{a_h}^n]\rangle_n = \langle[c_{a_h}n^{e_{a_h}}](\Sigma_{a_h'}c_{a_h'}n^{e_{a_h'}})^{-1}\rangle_n$. Intuitively, this is clear: at each n, the divisor $\Sigma_{a_h'}c_{a_h'}n^{e_{a_h'}}$ normalizes the vector so that it sums to one. Unfortunately, the details are nontrivial since the normalizing divisors must be carried into the expressions determining beliefs, and

such expressions have zero-limit denominators at every zero-probability information set.

PROOF 4.10 (for Theorem 4.1(b⇒c)). Assume (b). Define $(\langle[\pi_{x_h}^n]\rangle_n)_h$ by

$$(34) \qquad (\forall h)(\forall n) \ [\pi_{x_h}^n] = [c_{x_h} n^{e_{x_h}}]\cdot(\Sigma_{x_h'} c_{x_h'} n^{e_{x_h'}})^{-1} \ .$$

First Half. Note

$$(35) \qquad (\forall h) \quad \lim_{n\to\infty}\Sigma_{a_h} c_{a_h} n^{e_{a_h}}$$
$$=_1 \ \Sigma_{a_h}\lim_{n\to\infty} c_{a_h} n^{e_{a_h}}$$
$$=_2 \ \Sigma_{x_h}\pi_{a_h}$$
$$=_3 \ 1 \ ,$$

where $=_1$ holds by the first half of (b) and the algebra of limits, $=_2$ holds by the first half of (b), and $=_3$ holds by the well-definition of π. Then

$$(36) \qquad (\forall h) \quad [\pi_{a_h}]$$
$$=_1 \ \lim_{n\to\infty} [c_{a_h} n^{e_{a_h}}]$$
$$=_2 \ \lim_{n\to\infty} [c_{a_h} n^{e_{a_h}}](\lim_{n\to\infty}\Sigma_{x_h'} c_{x_h'} n^{e_{x_h'}})^{-1}$$
$$=_3 \ \lim_{n\to\infty} [c_{a_h} n^{e_{a_h}}](\Sigma_{x_h'} c_{x_h'} n^{e_{x_h'}})^{-1}$$
$$=_4 \ \lim_{n\to\infty} [\pi_{a_h}^n] \ ,$$

where $=_1$ holds by the first half of (b), $=_2$ holds by (35), $=_3$ holds by the first half of (b), (35) and the algebra of limits, and $=_4$ holds by the definition (34) of $\langle[\pi_{a_h}^n]\rangle_n$.

Second Half. Fix h. Define

$$e_h = \max\{ \ \Sigma_{k=0}^{\ell(x_h)-1} e_{\alpha\circ p_k(x_h)} \ | \ x_h \ \} \ ,$$

and note that

$$(37a) \qquad (\forall x_h) \lim_{n\to\infty} n^{-e_h}\cdot\Pi_{k=0}^{\ell(x_h)-1} c_{\alpha\circ p_k(x_h)} n^{e_{\alpha\circ p_k(x_h)}} \in [0, \infty) \text{ and}$$

$$(37b) \qquad (\exists x_h) \lim_{n\to\infty} n^{-e_h}\cdot\Pi_{k=0}^{\ell(x_h)-1} c_{\alpha\circ p_k(x_h)} n^{e_{\alpha\circ p_k(x_h)}} \in (0, \infty) \ .$$

Further, for each x_h, define Y_{x_h} by

$$Y_{x_h} = \{ \ \eta \ | \ (\exists a_\eta)(\exists k\in\{0, 1, \dots \ell(x)-1\}) \ a_\eta = \alpha\circ p_k(x_h) \ \}$$

(thus Y_{x_h} is the set of information sets that are passed through on the way to x_h.) Note that

$$(38) \qquad (\forall x_h) \quad \lim_{n\to\infty} n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1} c_{\alpha\circ p_k(x_h)} n^{e_{\alpha\circ p_k(x_h)}}$$

$$=_1 \frac{\lim_{n\to\infty} n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}c_{\alpha\circ p_k(x_h)}n^{e_{\alpha\circ p_k(x_h)}}}{\Pi_{\eta\in Y_{x_h}}\lim_{n\to\infty}\Sigma_{a_\eta}c_{a_\eta}n^{e_{a_\eta}}}$$

$$=_2 \lim_{n\to\infty}\frac{n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}c_{\alpha\circ p_k(x_h)}n^{e_{\alpha\circ p_k(x_h)}}}{\Pi_{\eta\in Y_{x_h}}\Sigma_{a_\eta}c_{a_\eta}n^{e_{a_\eta}}}$$

$$=_3 \lim_{n\to\infty} n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}\pi_{\alpha\circ p_k(x_h)}^n \;,$$

where $=_1$ holds by (35), $=_2$ holds by (35), (37a), and the algebra of limits, and $=_3$ holds by algebra and (34). Then

$$(\forall x_h)\quad \mu_{x_h}$$

$$=_1 \lim_{n\to\infty}\frac{\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}c_{\alpha\circ p_k(x_h)}n^{e_{\alpha\circ p_k(x_h)}}}{\Sigma_{x_h'}\,\rho_{p_{\ell(x_h')}(x_h')}\cdot\Pi_{k=0}^{\ell(x_h')-1}c_{\alpha\circ p_k(x_h')}n^{e_{\alpha\circ p_k(x_h')}}}$$

$$=_2 \lim_{n\to\infty}\frac{n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}c_{\alpha\circ p_k(x_h)}n^{e_{\alpha\circ p_k(x_h)}}}{\Sigma_{x_h'}\,n^{-e_h}\cdot\rho_{p_{\ell(x_h')}(x_h')}\cdot\Pi_{k=0}^{\ell(x_h')-1}c_{\alpha\circ p_k(x_h')}n^{e_{\alpha\circ p_k(x_h')}}}$$

$$=_3 \frac{\lim_{n\to\infty} n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}c_{\alpha\circ p_k(x_h)}n^{e_{\alpha\circ p_k(x_h)}}}{\Sigma_{x_h'}\lim_{n\to\infty} n^{-e_h}\cdot\rho_{p_{\ell(x_h')}(x_h')}\cdot\Pi_{k=0}^{\ell(x_h')-1}c_{\alpha\circ p_k(x_h')}n^{e_{\alpha\circ p_k(x_h')}}}$$

$$=_4 \frac{\lim_{n\to\infty} n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}\pi_{\alpha\circ p_k(x_h)}^n}{\Sigma_{x_h'}\lim_{n\to\infty} n^{-e_h}\cdot\rho_{p_{\ell(x_h')}(x_h')}\cdot\Pi_{k=0}^{\ell(x_h)-1}\pi_{\alpha\circ p_k(x_h')}^n}$$

$$=_5 \lim_{n\to\infty}\frac{n^{-e_h}\cdot\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}\pi_{\alpha\circ p_k(x_h)}^n}{\Sigma_{x_h'}\,n^{-e_h}\cdot\rho_{p_{\ell(x_h')}(x_h')}\cdot\Pi_{k=0}^{\ell(x_h)-1}\pi_{\alpha\circ p_k(x_h')}^n}\;,$$

$$=_6 \lim_{n\to\infty}\frac{\rho_{p_{\ell(x_h)}(x_h)}\cdot\Pi_{k=0}^{\ell(x_h)-1}\pi_{\alpha\circ p_k(x_h)}^n}{\Sigma_{x_h'}\,\rho_{p_{\ell(x_h')}(x_h')}\cdot\Pi_{k=0}^{\ell(x_h)-1}\pi_{\alpha\circ p_k(x_h')}^n}\;,$$

where $=_1$ holds by the second half of (b), $=_2$ holds by algebra, $=_3$ holds by (37) and the algebra of limits, $=_4$ holds by (38), $=_5$ holds (37), (38) and the algebra of limits, and $=_6$ holds by algebra. The entire equality is the second half of (c). $\qquad\square$

Appendix A. Real Exponents

Throughout the paper, the symbol e assumes integer values, and accordingly, Theorem 2.1 and Theorem 4.1(a&b) are concerned with characterizing consistency by means of integer exponents. This appendix notes that this paper's results for integer exponents are stronger than

analogous results for real exponents. In particular, Corollary A.1 follows from Theorem 4.1 and two components of its proof. Here $([\dot{e}_{a_h}])_h$ lists a vector $[\dot{e}_{a_h}]$ of real exponents $\dot{e}_{a_h} \in \mathbb{R}$ at every information set h.

COROLLARY A.1. *Let* $[T, \prec, A, \alpha, \rho, H]$ *be a game form. Then the following are equivalent for any assessment* $([\mu_{x_h}], [\pi_{a_h}])_h$. (*à*) *There exists a* $([c_{a_h} n^{\dot{e}_{a_h}}])_h$ *such that*

$$(\forall h)\ [\pi_{a_h}]\ is\ induced\ by\ [c_{a_h} n^{\dot{e}_{a_h}}]\ and$$

$$(\forall h)\ [\mu_{x_h}]\ is\ induced\ by\ \Sigma_{w(a_h)_h \in S_{x_h}}\ \rho_w \Pi_h c_{a_h} n^{\dot{e}_{a_h}}\ .$$

(*ḃ*) *There exists a* $([c_{a_h} n^{\dot{e}_{a_h}}])_h$ *such that*

$$(\forall h)\ [\pi_{a_h}] = lim_{n\to\infty}[c_{a_h} n^{\dot{e}_{a_h}}]\ and$$

$$(\forall h)\ [\mu_{x_h}]\ is\ induced\ by\ [\rho_{\ell(x_h)} \cdot \Pi_{k=0}^{\ell(x_h)-1} c_{\alpha \circ p_n(x_h)} n^{\dot{e}_{\alpha \circ p_n(x_h)}}]\ .$$

(*c*) $([\pi_{x_h}], [\mu_{x_h}])_h$ *is consistent.*

Proof. As with Theorem 4.1, the downhill implications are relatively easy. (à) implies (ḃ) by Proof 4.9 after replacing (a) with (à), (b) with (ḃ), and e with $\dot{e}$. Then, (ḃ) implies (c) by Proof 4.10 after replacing (b) with (ḃ) and e with $\dot{e}$.

(c) implies (à) via two steps. First, (c) implies statement (a) of Theorem 4.1 by Theorem 4.1 (this is the hard part). Second, this (a) implies (à) since the existence of monomials with integer coefficients implies the existence of monomials with real coefficients.   □

Theorem 4.1(a⇔b⇔c) is strictly stronger than Corollary A.1 to the extent that (a) is strictly stronger than (à) and to the extent that (b) is strictly stronger than (ḃ). In other words, Corollary A.1 derives the existence of real, but not necessarily integer, exponents. Corollary A.1(ḃ⇔c) appears to be equivalent to a reformulation of Theorem 3.1 in Perea y Monsuwe, Jansen, and Peters (1997).

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