Products of Several Relative Probabilities

by

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PRODUCTS OF SEVERAL RELATIVE PROBABILITIES

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Abstract. This paper defines and develops the concept of a product dispersion over any finite number of dimensions. The concept itself is nontrivial because products over several dimensions cannot be constructed by an iterative binary operation. Yet the paper's most important contribution is to characterize product dispersions by means of monomials. This result is derived through elementary linear algebra, and can be used to characterize the consistency of beliefs in extensive-form games (Streufert (2006a)).

1. Introduction

A dispersion is a system of relative probabilities over some underlying set. Such a dispersion must satisfy a basic cancellation law that resembles transitivity. If the underlying set is a Cartesian product, one can also define a stronger concept known as producthood. A product dispersion must satisfy not only the basic cancellation law, but also a vast number of other cancellation laws which embody the notion that cancellations can occur in the different dimensions independently. (Essentially, the concept of producthood extends the concept of independence from ordinary probabilities to relative probabilities.)

Theorem 5.1 is this paper's central result. It shows that a table of relative probabilities is a product iff it can be represented by a product of monomial vectors.

It might be useful to frame this result with an analogy. Consider consumer theory under the very restrictive assumption that the commodity space is finite. There we know that a binary relation is an ordering iff it can be represented by a utility function. Analogously, a table of relative probabilities is a dispersion iff it can be represented by

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a vector of monomials (McLennan (1989b) as reformulated in Streufert (2005, paragraph containing Note 4)).

Now continue this analogy when the underlying set is a Cartesian product. In consumer theory, we know that an ordering is separable across the dimensions of the Cartesian product iff it can be represented by a sum of (sub)utility functions defined at each dimension (as in Gorman (1968)). Analogously, Theorem 5.1 shows that a dispersion is a product iff it can be represented by a product of monomial vectors defined at each dimension.

Streufert (2006a) uses this theorem to characterize the consistency of beliefs in extensive-form games. Further, Streufert (2006b) shows that the absence of this theorem corresponds to a fallacy in the proofs of Kreps and Wilson (1982).

Although it takes nine pages to prove this theorem, the argument is surprisingly elementary. It draws on the insights of Scott (1964) and Krantz, Luce, Suppes, and Tversky (1971) and uses nothing more than basic linear algebra (the first paragraphs in Subsections 5.3 and 5.4 provide further details).

Although Theorem 5.1 is this paper's main contribution, a number of subsidiary results are also provided.

Section 2 develops an alternative formulation of producthood that allows one to use reciprocals. Section 3 shows that producthood coincides with dispersionhood when the underlying set is taken to be one-dimensional. Section 4 discusses the nontrivial relationship between a product and its marginals. And finally, Appendix A exhibits an example which demonstrates that three-dimensional products cannot be constructed by iteratively applying the two-dimensional concept.

All of the above employs nothing more than basic linear algebra. Section 6 mixes in a little topology. It shows that the set of products is compact (Theorem 6.1) and that it coincides with the closure of the set of positive products (Remark 6.3). Remark 6.3 is equivalent to a reformulation of a theorem in Kohlberg and Reny (1997) (and this result appears to be this paper's closest predecessor).

2. Products

2.1. Definition

This section formally defines producthood over several dimensions. For introductory examples and intuition in a two-dimensional setting, see Streufert (2005, Sections 2 and 3).

Let $(X_i)_{i=1}^{\ell}$ be a nonempty finite vector of nonempty finite sets (it is convenient but not logically necessary to make these ℓ sets disjoint). Then construct the Cartesian product $X = \Pi_{i=1}^{\ell} X_i$ containing vectors $x = (x_i)_{i=1}^{\ell}$. For example, if $X_1 = \{A, B, C\}$ and $X_2 = \{a, b\}$, then one of the six vectors in $X = X_1 \times X_2 = \{A, B, C\} \times \{a, b\}$ would be $x = x_1 x_2 = Cb$ (the notation $x = (x_1, x_2) = (C, b)$ is too clumsy).

Then let a *table* over X be a $[q_{x/x'}] \in [0, \infty]^{X^2}$ which lists a relative probability $q_{x/x'} \in [0, \infty]$ for every pair of elements x and x' from X. For example, suppose $X_1 = \{A, B, C\}$ and $X_2 = \{a, b\}$. Then the table $[q_{x/x'}]$ would contain $36 = |X|^2$ scalars, the scalar $q_{Cb/Aa}$ would give the probability of Cb relative to Aa, and the value $q_{Cb/Aa} = \infty$ would mean that Cb is infinitely more likely than Aa.

This paragraph uses an example to introduce the concept of a cancellation law. Imagine encountering the following product of three relative probabilities

$$q_{Cb/Ab} \times q_{Aa/Cb} \times q_{Bb/Ba} \ .$$

It would seem natural to cancel out the B in the "numerator" of the third term with the B in the "denominator" of that same term to arrive at

$$q_{Cb/Ab} \times q_{Aa/Cb} \times q_{\not{B}b/\not{B}a} \ ,$$

then to cancel out a pair of A's and a pair of C's to arrive at

$$q_{\not{C}b/\not{A}b} \times q_{\not{A}a/\not{C}b} \times q_{\not{B}b/\not{B}a} \ ,$$

and finally to cancel out a pair of a's and two pairs of b's to reach the conclusion that

$$q_{Cb/Ab} \times q_{Aa/Cb} \times q_{Bb/Ba} = 1 \ .$$

However, relative probabilities can take on values of 0 and ∞, and this can lead to cases in which our product of three terms is undefined. Accordingly, the formal definition of producthood below will impose the cancellation law

$$1 \in \odot(q_{Cb/Ab}, q_{Aa/Cb}, q_{Bb/Ba}) \ , \tag{1}$$

in which the set $\odot(q_{Cb/Ab}, q_{Aa/Cb}, q_{Bb/Ba})$ is either $[0, \infty]$ (in the case that one of $\{q_{Cb/Ab}, q_{Aa/Cb}, q_{Bb/Ba}\}$ is 0 and another is ∞) or the singleton containing the ordinary product $q_{Cb/Ab} \times q_{Aa/Cb} \times q_{Bb/Ba}$ (in all other cases). Hence (1) imposes a restriction exactly when the ordinary product is well-defined.

To formulate such a cancellation law rigorously, we must first define exactly what the symbol $\odot$ means. It is a set-valued function which assigns a subset of $[0,\infty]$ to every finite vector $(u^j)_{j=0}^m \in [0,\infty]^{1+m}$ of scalars u^j from $[0,\infty]$. It is defined by the rule

$$\odot(u^j)_{j=0}^m = \left(\begin{array}{cc} [0,\infty] & \text{if } (\exists j) u^j{=}0 \text{ and } (\exists j) u^j{=}\infty \\ \{\Pi_{j=0}^m u^j\} & \text{otherwise} \end{array} \right)$$

when $(u^j)_{j=0}^m$ is nonempty, and by the rule that the empty vector is mapped to $\{1\}$ (for example, $\odot(u^j)_{j=1}^0 = \{1\}$).

Here are two tangential remarks about $\odot$. First, the argument of $\odot$ is a vector and not a set. If it were a set, $\odot$ would need to assume the same value at $\{3\}$ as at $\{3,3\}$ and we do mean to say that $\odot(3) = \{3\}$ and $\odot(3,3) = \{9\}$. Second, the notation $\odot_{j=0}^m u^j$ is not used because it suggests that there is a binary relation $\odot$ which could derive $\odot(3,\infty,0)$ as $3\odot(\infty\odot 0)$. This cannot be done because $3\odot(\infty\odot 0)$ reduces to $3\odot[0,\infty]$, which is an ill-defined "product" of a scalar with a set.

Formally, a *cancellation law* is a statement of the form

$$(2) \qquad (\forall (x^j)_{j=0}^m)\ 1 \in \odot(q_{x^{\sigma,j}/x^j})_{j=0}^m\ ,$$

where m is a nonnegative scalar, where $\sigma = (\sigma_i)_{i=1}^\ell$ is a vector listing in each dimension i a permutation σ_i of the set $\{0,1,2,...\,m\}$, and where each vector $x^{\sigma,j} \in X$ is defined by

$$x^{\sigma,j} = (x_1^{\sigma_1(j)}, x_2^{\sigma_2(j)}, \dots x_n^{\sigma_n(j)})\ .$$

The integer m is called the *order* of the cancellation law, and, an application of the law at a particular $(x^j)_{j=0}^m$ is called an *instance* of the law.

For example, (1) is the instance $x^0{=}Ab$, $x^1{=}Cb$, $x^2{=}Ba$ of the second-order cancellation law

$$(3) \qquad (\forall x^0, x^1, x^2)\ 1 \in \odot\{q_{x_1^1 x_2^1/x_1^0 x_2^0}, q_{x_1^0 x_2^2/x_1^1 x_2^1}, q_{x_1^2 x_2^0/x_1^2 x_2^2}\}\ .$$

This law has the form (2) when the permutation vector $\sigma = (\sigma_1, \sigma_2)$ defined by $\sigma_1(0) = 1$, $\sigma_1(1) = 0$, $\sigma_1(2) = 2$ and $\sigma_2(0) = 1$, $\sigma_2(1) = 2$, $\sigma_2(2) = 0$. To make these two observations about the cancellation law (3), keep your eyes on its superscripts and note that its denominators are fixed.

Before proceeding further, notice that the cancellations in the example are occurring independently in the two dimensions. One can see this in both (1) and (3). This independent cancellation is the heart of the definition of producthood.

A *product* over $(X_i)_{i=1}^\ell$ is an element of the set $\Delta(X_i)_{i=1}^\ell$ defined by

$$\begin{aligned}(4)\qquad & \{\ [q_{x/x'}] \in [0,\infty]^{X^2} \mid \\ & (\forall m)(\forall \sigma)(\forall (x^j)_{j=0}^m)\ 1 \in \odot(q_{x^{\sigma,j}/x^j})_{j=0}^m \ \}\ .\end{aligned}$$

That is, a product is a table $[q_{x/x'}]$ over the Cartesian product $X = \Pi_i X_i$ that satisfies every instance (as specified by $(x^j)_{j=0}^m$) of every cancellation law (as specified by σ) of every order $m \geq 0$.

There are $((1{+}m)!)^\ell$ cancellation laws of order m because there are $(1{+}m)!$ permutations in each of the ℓ dimensions. Further, there are $(\Pi_{i=1}^\ell |X_i|)^{1+m}$ instances of *each* cancellation law because there are $\Pi_{i=1}^\ell |X_i|$ vectors in X. These numbers increase astronomically with the dimension ℓ, the order m, and the sizes $(|X_i|)_{i=1}^\ell$ of the underlying sets.

2.2. An Alternative Formulation

Remark 2.1 gives an alternative formulation of producthood. Among other things, this result implies that the ℓ-dimensional concept of producthood defined in this paper is an extension of the two-dimensional concept of producthood defined in Streufert (2005) (equation (21) there coincides with (5) here when $\ell = 2$). Further, that paper's Note 2 explains that producthood is similar and equivalent to the acyclicity appearing Kohlberg and Reny (1997, Theorem 2.10), and additionally, that producthood is dissimilar but nonetheless equivalent to further concepts in McLennan (1989b), Blume, Brandenburger, and Dekel (1991), Hammond (1994), and Kohlberg and Reny (1997).

Remark 2.1. *The set $\Delta(X_i)_{i=1}^\ell$ of products over $(X_i)_{i=1}^\ell$ is equal to*

$$\begin{aligned}(5)\qquad & \{\ [q_{x/x'}] \in [0,\infty]^{X^2} \mid \\ & (\forall m)(\forall \sigma)(\forall (x^j)_{j=0}^m)\ q_{x^0/x^{\sigma,0}} \in \odot(q_{x^{\sigma,j}/x^j})_{j=1}^m \ \}\ .\end{aligned}$$

The remainder of this subsection proves Remark 2.1 with the assistance of the following lemma.

Lemma 2.2. *For any $(u^j)_{j=0}^m \in [0,\infty]^{1+m}$, we have that $1 \in \odot(u^j)_{j=0}^m$ iff $1/u^0 \in \odot(u^j)_{j=1}^m$.*

Proof. If $m = 0$, the two inclusions become $1 \in \{u^0\}$ and $1/u^0 \in \{1\}$, which are equivalent.

If $m \geq 1$, there are twelve cases defined by

$\odot(u^j)_{j=1}^m=[0,\infty]$	T	T	T
$\odot(u^j)_{j=1}^m=\{\infty\}$	T	F	F
$(\exists v\in(0,\infty))\ \odot(u^j)_{j=1}^m=\{v\}$	F	$\Leftrightarrow$	F
$\odot(u^j)_{j=1}^m=\{0\}$	F	F	T
	$u^0=0$	$u^0\in(0,\infty)$	$u^0=\infty$

.

In five cases I will show that both inclusions are true, in six cases I will show that both inclusions are false, and in the remaining case I will show that the two inclusions are equivalent by another route.

First consider the top three cases. There, $\odot(u^j)_{j=1}^m$ is $[0,\infty]$, thus $\odot(u^j)_{j=0}^m$ is also $[0,\infty]$, and thus, both inclusions vacuously hold.

Then consider those three of the remaining nine cases in which $u^0 = 0$. In the case $\odot(u^j)_{j=1}^m = \{\infty\}$, the left inclusion is true because $\odot(u^j)_{j=0}^m = [0,\infty]$ and the right inclusion is also true because $1/u^0 = \infty$. In the case $(\exists v\in(0,\infty))\ \odot(u^j)_{j=1}^m = \{v\}$ and also in the case $\odot(u^j)_{j=1}^m = \{0\}$, the left inclusion is false since $\odot(u^j)_{j=0}^m = \{0\}$ and the right inclusion is also false since $1/u^0 = \infty$.

Next consider those three of the remaining six cases in which $u^0 \in (0,\infty)$. In the case $\odot(u^j)_{j=1}^m = \{\infty\}$, the left inclusion fails since $\odot(u^j)_{j=0}^m = \{\infty\}$ and the right inclusion also fails since $1/u^0 \in (0,\infty)$. In the case $(\exists v\in(0,\infty))\ \odot(u^j)_{j=1}^m = \{v\}$, the left inclusion is equivalent to $1 = \Pi_{j=0}^m u^j$ which is equivalent to $1/u^0 = \Pi_{j=1}^m u^j$ which is equivalent to the right inclusion. In the case $\odot(u^j)_{j=1}^m = \{0\}$, the left inclusion fails because $\odot(u^j)_{j=0}^m = \{0\}$ and the right inclusion fails because $1/u^0 \in (0,\infty)$.

Finally consider the remaining three cases. Here $u^0 = \infty$. In the case $\odot(u^j)_{j=1}^m = \{\infty\}$ and also in the case $(\exists v\in(0,\infty))\ \odot(u^j)_{j=1}^m = \{v\}$, the left inclusion is false since $\odot(u^j)_{j=0}^m = \{\infty\}$ and the right inclusion is also false since $1/u^0 = 0$. In the case $\odot(u^j)_{j=1}^m = \{0\}$, the left inclusion is true because $\odot(u^j)_{j=0}^m = [0,\infty]$ and the right inclusion is also true since $1/u^0 = 0$. □

Proof of Remark 2.1. (4) $\subseteq$ (5). Suppose that $[q_{x/x'}]$ is an element of (4). A first-order cancellation law is $(\forall x, x')\ 1 \in \odot\{q_{x/x'}, q_{x'/x}\}$, and thus it must be the case that $(\forall x, x')\ q_{x/x'} = 1/q_{x'/x}$. Hence $[q_{x/x'}]$ is an element of (5) by Lemma 2.2.

(4) $\supseteq$ (5). Suppose that $[q_{x/x'}]$ is an element of the set (5). The zero-order cancellation law is $(\forall x)\ q_{x/x} \in \{1\}$ and a second-order cancellation law is $(\forall x, x')\ q_{x/x} \in \odot\{q_{x/x'}, q_{x'/x}\}$. By using the first inclusion to

replace the $q_{x/x}$ in the second, one finds that $(\forall x, x')\ 1 \in \odot\{q_{x/x'}, q_{x'/x}\}$, and thus, that $(\forall x, x')\ q_{x/x'} = 1/q_{x'/x}$. Hence $[q_{x/x'}]$ is an element of (4) by Lemma 2.2. □

3. Dispersions

3.1. Definition

Consider any finite set Z. Streufert (2005, page 9) defines a *dispersion* over Z to be a table $[q_{z/z'}]$ that satisfies *unit diagonality*

$$(\forall z)\ q_{z/z'} = 1 \tag{6}$$

and the *basic cancellation law*

$$(\forall z, z', z'')\ q_{z/z''} \in \odot(q_{z/z'}, q_{z'/z''})\ . \tag{7}$$

It is easy to show that every dispersion satisfies *reciprocity*

$$(\forall z, z')\ q_{z/z'} = 1/q_{z'/z} \tag{8}$$

(unit diagonality and the basic cancellation law at $(z, z', z'') = (z, z', z)$ imply that $(\forall z, z')\ 1 = q_{z/z} \in \odot(q_{z/z'}, q_{z'/z})$, and thus, it must be the case that either $q_{z/z'}q_{z'/z} = 1$ for some positive finite numbers $q_{z/z'}$ and $q_{z'/z}$, or that one of $q_{z/z'}$ and $q_{z'/z}$ is zero and the other is infinity).

Note 1 of that Streufert (2005) explains that a dispersion here is similar and equivalent to a matrix of log-likelihoods in McLennan (1989b), a conditional probability system in Myerson (1986), and a random variable defined on a relatively probability space in Kohlberg and Reny (1997). The same note also explains how dispersionhood is dissimilar but nonetheless equivalent to concepts in McLennan (1989a), Blume, Brandenburger, and Dekel (1991), Monderer, Samet, and Shapley (1992), Hammond (1994), and Vieille (1996).

3.2. Dispersionhood is One-Dimensional Producthood

Note that Section 2's definition of producthood can be applied to one set (i.e., to a one-dimensional vector of sets). Accordingly, the set $\Delta(Z)$ of "products" over some set Z is

$$\begin{gathered}\{\ [q_{z/z'}] \in [0,\infty]^{Z^2}\ | \\ (\forall m)(\forall \sigma)(\forall (z^j)_{j=0}^m)\ 1 \in \odot(q_{z^{\tau(j)}/z^j})_{j=0}^m\ \}\ ,\end{gathered} \tag{9}$$

where τ is a permutation of $\{0, 1, ... m\}$. In particular, (9) is the definition (4) of producthood evaluated at $\ell = 1$, $X_1 = Z$, and $\sigma_1 = \tau$.

Remark 3.1 shows that $\Delta(Z)$ equals the set of dispersions over Z. Accordingly, one-dimensional producthood is equivalent to dispersionhood (and hence $\Delta(Z)$ can be used to denote the set of all dispersions over Z).

Remark 3.1. *For any set Z, $\Delta(Z)$ is the set of dispersions over Z.*

Proof. Necessity of Dispersionhood. By Remark 2.1, the set (9) is equal to

$$\text{(10)} \qquad \{\ [q_{z/z'}] \in [0,\infty]^{Z^2} \mid (\forall m)(\forall\tau)(\forall(z^j)_{j=0}^m)\ q_{z^0/z^{\tau(0)}} \in \odot(q_{z^{\tau(j)}/z^j})_{j=1}^m\ \}$$

Since the only permutation of $\{0\}$ is the identity map, (10)'s cancellation law at $m = 0$ is

$$(\forall z^0)\ q_{z^0/z^0} \in \{1\},$$

which is equivalent to unit diagonality (6). Further, (10)'s cancellation law at $m = 2$ and the permutation $\tau(0) = 2$, $\tau(1) = 0$, $\tau(2) = 1$ is

$$(\forall z^0, z^1, z^2)\ q_{z^0/z^2} \in \odot(q_{z^0/z^1}, q_{z^1/z^2})\ ,$$

which is the basic cancellation law (7).

Sufficiency of Dispersionhood. Suppose that $[q_{z/z'}]$ is a dispersion. Then fix any $m \geq 0$, any permutation τ of $\{0, 1, ... m\}$, and any $(z^j)_{j=0}^m$. Our task is to derive (9)'s cancellation law, namely, $1 \in \odot(q_{z^{\tau(j)}/z^j})_{j=0}^m$. Since this holds trivially if $(\exists j)\ q_{z^{\tau(j)}/z^j} = 0$ and $(\exists j)\ q_{z^{\tau(j)}/z^j} = \infty$, we may assume without loss of generality that either every element of $(q_{z^{\tau(j)}/z^j})_{j=0}^m$ is finite or every element is positive.

First suppose that every element of $(q_{z^{\tau(j)}/z^j})_{j=0}^m$ is finite. This and the next two paragraphs will show that for all $k \in \{0, 1, ... m\}$, there exists a permutation τ^k of $\{0, 1, ... k\}$ such that

$$\text{(11a)} \qquad \Pi_{j=0}^m q_{z^{\tau(j)}/z^j} = \Pi_{j=0}^k q_{z^{\tau^k(j)}/z^j} \text{ and}$$

$$\text{(11b)} \qquad \text{every element of } (q_{z^{\tau^k(j)}/z^j})_{j=0}^k \text{ is finite.}$$

This task will be accomplished by induction on k. The initial step at $k = m$ holds by defining $\tau^m = \tau$: (11a) is then trivial and (11b) holds by the assumption beginning this paragraph. Now suppose there is a τ^k satisfying (11) at some $k \in \{1, 2, ... m\}$. The next two paragraphs derive a τ^{k-1} satisfying (11) at $k-1$. Two cases arise.

On the one hand, it might be that $\tau^k(k) = k$. In this case, let τ^{k-1} be the restriction of τ^k to $\{0, 1, ... k-1\}$. Then (11b) at $k-1$ follows from (11b) at k. Further (11a) at $k{-}1$ holds by

$$\begin{aligned} & \Pi_{j=0}^{m} q_{z^{\tau^k(j)}/z^j} \\ =_1\ & \Pi_{j=0}^{k} q_{z^{\tau^k(j)}/z^j} \\ =_2\ & (\Pi_{j=0}^{k-1} q_{z^{\tau^k(j)}/z^j}) \times q_{z^{\tau^k(k)}/z^k} \\ =_3\ & (\Pi_{j=0}^{k-1} q_{z^{\tau^{k-1}(j)}/z^j}) \times q_{z^k/z^k} \\ =_4\ & \Pi_{j=0}^{k-1} q_{z^{\tau^{k-1}(j)}/z^j}\ , \end{aligned}$$

where $=_1$ holds by the (11a) at k, $=_3$ holds by the definition of τ^{k-1} and the case definition $\tau^k(k) = k$, and $=_4$ holds by the unit diagonality (6) of the dispersion $[q_{z/z'}]$.

On the other hand, it might be that $\tau^k(k) \neq k$. In this case, let $k^0 = (\tau^k)^{-1}(k)$, and note that $k^0 \neq k$ because τ is a permutation and $\tau^k(k) \neq k$. Then define τ^{k-1} at k^0 to be $\tau^k(k)$, and define τ^{k-1} over the remainder of $\{0, 1, ...\, k{-}1\}$ to coincide with τ^k. Casually speaking, τ^{k-1} is defined so as to "bridge" over k by mapping k's "predecessor" k^0 to k's "successor" $\tau^k(k)$. We begin with two observations about the probability of k's successor relative to k's predecessor:

(12a)
$$\begin{aligned} & q_{z^{\tau^{k-1}(k^0)}/z^{k^0}} \\ =_1\ & q_{z^{\tau^k(k)}/z^{k^0}} \\ =_2\ & q_{z^{\tau^k(k)}/z^k} \times q_{z^k/z^{k^0}}\ , \end{aligned}$$

(12b) and $q_{z^{\tau^{k-1}(k^0)}/z^{k^0}}$ is finite,

where the $=_1$ holds by the definition of τ^{k-1}, and both $=_2$ and (12b) hold by the basic cancellation law (7) of the dispersion $[q_{z/z'}]$ and by the fact that the two factors after $=_2$ are finite by (11b) at k. Then (11b) at $k{-}1$ holds by (12b), by (11b) at k, and by the definition of τ^{k-1}. Further (11a) holds at $k{-}1$ by

$$\begin{aligned} & \Pi_{j=0}^{m} q_{z^{\tau^k(j)}/z^j} \\ =_1\ & \Pi_{j=0}^{k} q_{z^{\tau^k(j)}/z^j} \\ =_2\ & (\Pi_{j\in\{0,1,\dots\,k\}\sim\{k,k^0\}} q_{z^{\tau^k(j)}/z^j}) \times q_{z^{\tau^k(k)}/z^k} \times q_{z^k/z^{k^0}} \\ =_3\ & (\Pi_{j\in\{0,1,\dots\,k\}\sim\{k,k^0\}} q_{z^{\tau^k(j)}/z^j}) \times q_{z^{\tau^{k-1}(k^0)}/z^{k^0}} \end{aligned}$$

$$=_4 \ (\Pi_{j\in\{0,1,\ldots k-1\}\sim\{k^0\}} q_{z^{\tau^{k-1}(j)}/z^j}) \times q_{z^{\tau^{k-1}(k^0)}/z^{k^0}}$$
$$=_5 \ \Pi_{j=0}^{k-1} q_{z^{\tau^{k-1}(j)}/z^j} \ ,$$

where $=_1$ holds by (11a) at k, $=_2$ holds by the definition of k^0, $=_3$ holds by (12a), and $=_4$ holds by the definition of τ^{k-1}.

Thus, we may conclude that there is a permutation τ^0 of $\{0\}$ such that (11a) holds at $k = 0$. In other words, there is some τ^0 such that

$$\Pi_{j=0}^{m} q_{z^{\tau(j)}/z^j} = \Pi_{j=0}^{0} q_{z^{\tau^0(0)}/z^0} \ .$$

Therefore, since the only permutation of $\{0\}$ is the identity map and since $[q_{z/z'}]$ has a unit diagonal (6) by dispersionhood,

$$\Pi_{j=0}^{m} q_{z^{\tau(j)}/z^j} = q_{z^0/z^0} = 1 \ .$$

Recall the two cases laid out in the second paragraph of the proof. The first assumed that every element of $(q_{z^{\tau(j)}/z^j})_{j=0}^m$ was finite. Now assume that every element of $(q_{z^{\tau(j)}/z^j})_{j=0}^m$ is positive. Here

$$\begin{aligned}
&\odot(q_{z^{\tau(j)}/z^j})_{j=0}^m \\
=_1 \ & \Pi_{j=0}^m q_{z^{\tau(j)}/z^j} \\
=_2 \ & [\Pi_{j=0}^m q_{z^j/z^{\tau(j)}}]^{-1} \\
=_3 \ & [\Pi_{j=0}^m q_{z^{\tau^{-1}(j)}/z^j}]^{-1} \\
=_4 \ & [1]^{-1} = 1 \ ,
\end{aligned}$$

where $=_1$ follows from the positivity of every $q_{z^{\tau(j)}/z^j}$, $=_2$ follows from reciprocity (8), $=_3$ holds because multiplication is commutative, and $=_4$ follows from the previous case since every $q_{z^{\tau^{-1}(j)}/z^j}$ is finite because every $q_{z^{\tau(j)}/z^j}$ is positive. □

3.3. Every Product is a Dispersion

Now consider a vector of sets $(X_i)_{i=1}^\ell$. By (4) in Section 2, the symbol $\Delta(X_i)_{i=1}^\ell$ denotes the set of products over $(X_i)_{i=1}^\ell$. By (9) in this section, the symbol $\Delta(\Pi_{i=1}^\ell X_i)$ denotes the set of dispersions over $\Pi_{i=1}^\ell X_i$. These two sets are comparable because both the products and the dispersions are tables over $\Pi_{i=1}^\ell X_i$. Remark 3.2 shows that every product is a dispersion (and thus the terms "product" and "product dispersion" are synonymous).

Remark 3.2. *$\Delta(X_i)_{i=1}^\ell$ is a subset of $\Delta(\Pi_{i=1}^\ell X_i)$.*

Proof. As in (4), let $X = \Pi_{i=1}^{\ell}$, let x denote a vector in X, and let σ denote a vector of ℓ permutations. Also, as in (9), let τ denote one permutation. Then

$$\begin{aligned}
&\Delta(X_i)_{i=1}^{\ell} \\
=_1\ & \{\ [q_{x/x'}] \mid (\forall m)(\forall \sigma)(\forall (x^j)_{j=0}^{m})\ 1 \in \odot(q_{x^{\sigma,j}/x^j})_{j=0}^{m}\ \} \\
\subseteq_2\ & \{\ [q_{x/x'}] \mid (\forall m)(\forall \sigma)\ \sigma_1 = \sigma_2 = \cdots = \sigma_\ell \text{ implies} \\
& \qquad (\forall (x^j)_{j=0}^{m})\ 1 \in \odot(q_{x^{\sigma,j}/x^j})_{j=0}^{m}\ \} \\
=_3\ & \{\ [q_{x/x'}] \mid (\forall m)(\forall \tau)(\forall (x^j)_{j=0}^{m})\ 1 \in \odot(q_{x^{\tau(j)}/x^j})_{j=0}^{m}\ \} \\
=_4\ & \Delta(\Pi_{i=1}^{\ell} X_i)\ ,
\end{aligned}$$

where $=_1$ is (4), $=_3$ follows from setting $\tau = \sigma_1$, and $=_4$ is (9). □

Further, Remark 3.2's inclusion is typically strict because $\subseteq_2$ typically holds only one way. In other words, dispersions are typically not products because products satisfy the cancellation law for all permutation vectors $\sigma = (\sigma_1, \sigma_2, \ldots \sigma_\ell)$, while dispersions need only satisfy the cancellation law for permutation vectors in which $\sigma_1 = \sigma_2 = \cdots = \sigma_\ell$. Essentially, the definition of dispersionhood over $\Pi_{i=1}^{\ell} X_i$ treats the elements of $\Pi_{i=1}^{\ell} X_i$ as if they were one-dimensional.

In accord with the preceding paragraph, there are instances in which $\Delta(\Pi_{i=1}^{\ell} X_i) \sim \Delta(X_i)_{i=1}^{\ell}$ is nonempty. In other words, there are dispersions over Cartesian products which fail to be products. One such example is discussed in Streufert (2003, next-to-last paragraph of Section 3.2): it has $\ell = 2$ and $|X_1| = |X_2| = 3$ and reformulates Kohlberg and Reny (1997, Figure 1). A second such example appears in the proof of this paper's Remark A.1(b): it has $\ell = 3$ and $|X_1| = |X_2| = |X_3| = 2$, it is a dispersion over $\Pi_{i=1}^{3} X_i$ because it is a product over $(X_1, X_2 \times X_3)$, and yet, it is not a product over (X_1, X_2, X_3). I am not aware of any simpler examples.

4. Marginals

The *marginals* of a product $[q_{x/x'}]$ over $(X_i)_{i=1}^{\ell}$ are the ℓ dispersions $([q_{x_i/x_i'}])_{i=1}^{\ell}$ which satisfy

$$(\forall x, x')\ q_{x/x'} \in \odot(q_{x_i/x_i'})_{i=1}^{\ell}\ . \tag{13}$$

Note that marginals are defined to be dispersions, and consequently, each marginal must itself satisfy unit diagonality (6) and the basic cancellation law (7) (hence "marginal" and "marginal dispersion" are synonymous).

Remark 4.1. *Every product* $[q_{x/x'}]$ *over* $(X_i)_{i=1}^{\ell}$ *has a unique vector of marginals* $([q_{x_i/x_i'}])_{i=1}^{\ell}$. *Further, for any dimension* i, *the marginal* $[q_{x_i/x_i'}]$ *equals* $[q_{x_i x_{-i}^\star/x_i' x_{-i}^\star}]$ *for any* $x_{-i}^\star \in \Pi_{j\neq i}X_j$.

Proof. Take any product $[q_{x/x'}]$ over $(X_i)_{i=1}^{\ell}$.

First we show that $[q_{x/x'}]$ has at least one vector of marginals. Choose some x° and define each $[q_{x_i/x_i'}]$ to be $[q_{x_i x_{-i}^\circ/x_i' x_{-i}^\circ}]$. Then (13) holds because

$$(\forall x, x')\ q_{x/x'} \in_1 \odot(q_{x_i x_{-i}^\circ/x_i' x_{-i}^\circ})_{i=1}^{\ell} =_2 \odot(q_{x_i/x_i'})_{i=1}^{\ell}\ ,$$

where $\in_1$ is an ℓth-order cancellation law (this can be verified by canceling terms) and $=_2$ holds by the definition of $([q_{x_i/x_{-i}}])_{i=1}^{\ell}$. Further, note that any restriction of a dispersion to a smaller domain is necessarily a dispersion in its own right (this holds because the satisfaction of (6) and (7) on the original domain implies their satisfaction on the smaller domain). Thus, each $[q_{x_i/x_i'}]$ is a dispersion because it is defined to be a restriction of the (product) dispersion $[q_{x/x'}]$.

Second, suppose that $([q_{x_i/x_i'}])_{i=1}^{\ell}$ is a vector of marginals. Thus

$$\begin{aligned}(\forall i)(\forall x_i, x_i', x_{-i}^\star)\quad & q_{x_i x_{-i}^\star/x_i' x_{-i}^\star} \\ & \in_1 \odot(q_{x_i/x_i'}, (q_{x_j^\star/x_j^\star})_{j\neq i}) \\ & =_2 \odot(q_{x_i/x_i'}, 1, 1, \ldots 1) \\ & =_3 \{q_{x_i/x_i'}\}\ . \end{aligned} \tag{14}$$

where $\in_1$ holds by (13) and $=_2$ holds by the unit diagonality of every marginal. Thus each marginal $[q_{x_i/x_i'}]$ must equal $[q_{x_i x_{-i}^\star/x_i' x_{-i}^\star}]$ for any value of $x_{-i}^\star$. □

Finally, start anew with a nonempty finite vector $(X_i)_{i=1}^{\ell}$ of nonempty finite sets, and a vector of dispersions $([q_{x_i/x_i'}])_{i=1}^{\ell}$ over those sets. As might well be expected, we define *a product of a vector of dispersions* $([q_{x_i/x_i'}])_{i=1}^{\ell}$ to be a product over $(X_i)_{i=1}^{\ell}$ whose marginals are $([q_{x_i/x_i'}])_{i=1}^{\ell}$.

However, keep in mind that the marginals of a product are unique (by Remark 4.1) but that the product of ℓ dispersions might not be unique (as illustrated by Section 4.6 of Streufert (2005)). This can be surprisingly tricky to digest because ordinary probability distributions are fundamentally different: there the ℓ marginals of a product distribution are unique, *and* the product of ℓ distributions is unique. Thus, a product distribution over ℓ variables is equivalent to its ℓ marginal distributions. Unfortunately, we don't have that luxury here: marginal dispersions are ambiguous.

5. Representation by Monomial Vectors

5.1. Theorem

In mathematics, a "monomial in the variable x" is an expression of the form cx^e, where the coefficient c is a real number and the exponent e is a nonnegative integer. Here a *monomial* will refer to any expression of the form cn^e, where c is a positive real number and e is an integer.

Consider a set Z. A vector $[c_z n^{e_z}]$ of monomials over Z is said to *represent* a table $[q_{z/z'}]$ over Z if

$$(15) \qquad (\forall z, z')\ q_{z/z'} = \begin{pmatrix} \infty & \text{if } e_z > e_{z'} \\ c_z/c_{z'} & \text{if } e_z = e_{z'} \\ 0 & \text{if } e_z < e_{z'} \end{pmatrix} .$$

Streufert (2005, note 4) slightly modifies McLennan (1989b, page 147) to show that a table over Z can be represented by a monomial vector $[c_z n^{e_z}]$ iff it is a dispersion. Analogously, Streufert (2005, Theorem 4.1) demonstrates that a table over $X \times Y$ can be represented by some $[c_x n^{e_x} \cdot c_y n^{e_y}]$ iff it is a product over (X, Y). The following theorem extends this two-dimensional result to ℓ dimensions.

Theorem 5.1. *Let $[q_{x/x'}]$ be a table over $\Pi_{i=1}^{\ell} X_i$. Then (a) $[q_{x/x'}]$ is represented by some $[\Pi_{i=1}^{\ell} c_{x_i} n^{e_{x_i}}]$ iff (b) $[q_{x/x'}]$ is a product over $(X_i)_{i=1}^{\ell}$. Further, the marginals of the product represented by $[\Pi_{i=1}^{\ell} c_{x_i} n^{e_{x_i}}]$ are represented by $([c_{x_i} n^{e_{x_i}}])_{i=1}^{\ell}$.*

Proof. (a) implies (b) by Proof 5.2 (in the next subsection). The converse holds by Proof 5.5 (this is the hard part: its derivation fills Subsections 5.3, 5.4, and 5.5). Finally, the theorem's second sentence holds by Proof 5.6 (in Subsection 5.6). □

Note that

$$(\forall x)\ \Pi_{i=1}^{\ell} c_{x_i} n^{e_{x_i}} = (\Pi_{i=1}^{\ell} c_{x_i}) n^{(\Sigma_{i=1}^{\ell} e_{x_i})} ,$$

and thus by definition (15), statement (a) in Theorem 5.1 is equivalent to the existence of $([c_{x_i}])_{i=1}^{\ell}$ and $([e_{x_i}])_{i=1}^{\ell}$ such that

$$(16) \quad (\forall x, x')\ q_{x/x'} = \begin{pmatrix} \infty & \text{if } \Sigma_{i=1}^{\ell} e_{x_i} > \Sigma_{i=1}^{\ell} e_{x'_i} \\ (\Pi_{i=1}^{\ell} c_{x_i})/(\Pi_{i=1}^{\ell} c_{x'_i}) & \text{if } \Sigma_{i=1}^{\ell} e_{x_i} = \Sigma_{i=1}^{\ell} e_{x'_i} \\ 0 & \text{if } \Sigma_{i=1}^{\ell} e_{x_i} < \Sigma_{i=1}^{\ell} e_{x'_i} \end{pmatrix} .$$

Appendix B notes that Theorem 5.1 is stronger than an analogous result with real as opposed to integer exponents.

5.2. Proof of Theorem 5.1($a \Rightarrow b$)

Proof 5.2. Suppose $[q_{x/x'}]$ satisfies (a), which by the observation at (16) is equivalent to the existence of $([c_{x_i}])_{i=1}^{\ell}$ and $([e_{x_i}])_{i=1}^{\ell}$ such that

$$(17)\quad (\forall x, x')\ q_{x/x'} = \begin{pmatrix} \infty & \text{if } \Sigma_{i=1}^{\ell} e_{x_i} > \Sigma_{i=1}^{\ell} e_{x'_i} \\ (\Pi_{i=1}^{\ell} c_{x_i})/(\Pi_{i=1}^{\ell} c_{x'_i}) & \text{if } \Sigma_{i=1}^{\ell} e_{x_i} = \Sigma_{i=1}^{\ell} e_{x'_i} \\ 0 & \text{if } \Sigma_{i=1}^{\ell} e_{x_i} < \Sigma_{i=1}^{\ell} e_{x'_i} \end{pmatrix} .$$

Our task is to show (b), which by definition (4) is equivalent to

$$(18)\qquad (\forall m)(\forall \sigma)(\forall (x^j)_{j=0}^m)\ 1 \in \odot(q_{x^{\sigma,j}/x^j})_{j=0}^m .$$

Accordingly, take any m, σ, and $(x^j)_{j=0}^m$. Note that

$$\begin{aligned} (\forall i)(\forall x_i) \quad & |\{\ j \mid x_i^{\sigma,j}{=}x_i\ \}| \\ &=_1 |\{\ j \mid x_i^{\sigma_i(j)}{=}x_i\ \}| \\ &=_2 |\{\ j \mid x_i^j{=}x_i\ \}| , \end{aligned}$$

where $=_1$ holds by the definition of $x^{\sigma,j}$ and $=_2$ holds by the fact that σ_i is a permutation. This observation yields

$$\begin{aligned} &(\forall i)\ \Pi_{j=0}^m c_{x_i^{\sigma,j}} = \Pi_{j=0}^m c_{x_i^j} \text{ and} \\ &(\forall i)\ \Sigma_{j=0}^m e_{x_i^{\sigma,j}} = \Sigma_{j=0}^m c_{x_i^j} , \end{aligned}$$

which in turn yields

$$(19a)\qquad \Pi_{j=0}^m \Pi_{i=1}^{\ell} c_{x_i^{\sigma,j}} = \Pi_{j=0}^m \Pi_{i=1}^{\ell} c_{x_i^j} \text{ and}$$

$$(19b)\qquad \Sigma_{j=0}^m \Sigma_{i=1}^{\ell} e_{x_i^{\sigma,j}} = \Sigma_{j=0}^m \Sigma_{i=1}^{\ell} e_{x_i^j} .$$

First suppose that there is a term j such that $\Sigma_{i=1}^{\ell} e_{x_i^{\sigma,j}} > \Sigma_{i=1}^{\ell} e_{x_i^j}$. Then (19b) implies there is another term j' such that $\Sigma_{i=1}^{\ell} e_{x_i^{\sigma,j'}} < \Sigma_{i=1}^{\ell} e_{x_i^{j'}}$. Hence by (17), $q_{x^{\sigma,j}/x^j} = \infty$ and $q_{x^{\sigma,j'}/x^{j'}} = 0$. Thus, (18) holds vacuously.

Second suppose that there is a term j such that $\Sigma_{i=1}^{\ell} e_{x_i^{\sigma,j}} < \Sigma_{i=1}^{\ell} e_{x_i^j}$. Then, by an argument symmetric to that of the previous paragraph, (18) holds vacuously.

Finally, suppose that every term j satisfies $\Sigma_{i=1}^{\ell} e_{x_i^{\sigma,j}} = \Sigma_{i=1}^{\ell} e_{x_i^j}$. Then by (17), we have

$$(20)\qquad (\forall j)\ q_{x^{\sigma,j}/x^j} = (\Pi_{i=1}^{\ell} c_{x_i^{\sigma,j}})/(\Pi_{i=1}^{\ell} c_{x^j}) .$$

Hence (18) holds by

$$\begin{aligned}
&\odot(\ q_{x^{\sigma,j}/x^j}\ )_{j=0}^m \\
=_1\ &\odot(\ (\Pi_{i=1}^{\ell} c_{x_i^{\sigma,j}})/(\Pi_{i=1}^{\ell} c_{x_i^j})\ )_{j=0}^m \\
=_2\ &\{\ \Pi_{j=0}^m\ (\Pi_{i=1}^{\ell} c_{x_i^{\sigma,j}})/(\Pi_{i=1}^{\ell} c_{x_i^j})\ \} \\
=_3\ &\{\ (\Pi_{j=0}^m \Pi_{i=1}^{\ell} c_{x_i^{\sigma,j}})/(\Pi_{j=0}^m \Pi_{i=1}^{\ell} c_{x_i^j})\ \} \\
=_4\ &\{1\}\ ,
\end{aligned}$$

where $=_1$ holds by (20), $=_2$ holds by the definition of $\odot$ and the fact that every coefficient c is a positive real, $=_3$ holds by algebra, and $=_4$ holds by (19a). ☐

5.3. An Outside Result

This and the next two subsections derive Theorem 5.1(b⇒a). These three subsections are the heart of the paper.

This subsection's Lemma 5.3 is used to derive exponents in the next subsection. It concerns linear algebra. In particular, it states that there is a solution to the system of linear inequalities and equalities in (21) precisely when the rows used to define those inequalities and equalities are "independent" in the sense of (22). This is analogous to the high-school-level result which states that there is a solution to $Ax = b$ if the rows of A are independent in the usual sense.

The lemma is a very minor variation on Krantz, Luce, Suppes, and Tversky (1971, Theorem 2.7). Since their proof depends only on high-school-level results for systems of linear equalities, and since the proof of Theorem 5.1 will depend only on their result, it is reasonable to say that the mathematics underneath Theorem 5.1 is elementary.

Lemma 5.3. *For any matrices $P \in \mathbb{Q}^{pk}$ and $A \in \mathbb{Q}^{ak}$, the following are equivalent.*

$$(21) \qquad (\exists w{\in}\mathbb{Z}^k)\ Pw \gg 0 \textit{ and } Aw = 0.$$

$$(22) \qquad \textit{Not } (\exists \pi{\in}\mathbb{Z}_+^p{\sim}\{0\})(\exists \alpha{\in}\mathbb{Z}^a)\ \pi^T P + \alpha^T A = 0.$$

($\mathbb{Q}$ denotes the set of rationals, $\mathbb{Z}$ denotes the set of integers, and $Pw \gg 0$ means that every element of the vector Pw is positive.)

Proof. Take any such P and A. The equivalence of (21) and (22) is equivalent to satisfying exactly one of the following.

$$(23) \qquad (\exists w{\in}\mathbb{Z}^k)\ Pw \gg 0 \text{ and } Aw = 0.$$

$$(24) \qquad (\exists \pi{\in}\mathbb{Z}_+^p{\sim}\{0\})(\exists \alpha{\in}\mathbb{Z}^a)\ \pi^T P + \alpha^T A = 0.$$

Krantz, Luce, Suppes, and Tversky (1971, Theorem 2.7 on page 62 and the first two sentences on page 63) yields that exactly one of the following must hold.

$$(\exists w{\in}\mathbb{Q}^k)\ Pw \gg 0 \text{ and } Aw = 0\ . \tag{25}$$

$$(\exists \pi{\in}\mathbb{Q}^p_+{\sim}\{0\})(\exists \alpha{\in}\mathbb{Q}^a)\ \pi^T P + \alpha^T A = 0 \text{ and } \pi{\cdot}1 = 1\ . \tag{26}$$

This follows from their result by replacing their m' with p, their m'' with a,

$$\text{their} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_{m'} \end{bmatrix} \text{with } P\ ,\ \text{their} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \ldots \\ \beta_{m''} \end{bmatrix} \text{with } -A\ ,$$

their x with w, their λ with π, and their μ with α.

As discussed informally by Krantz, Luce, Suppes, and Tversky (1971, page 63, sentences 3 through 6), (24) is equivalent to (26): (24) is implied by (26) by multiplying π and α by the product of all the denominators in these two vectors of rational numbers, and conversely, (24) implies (26) by dividing both π and α by $\pi{\cdot}1$. Similarly, (23) is equivalent to (25): (23) is implied by (25) by multiplying w by the product of all its denominators, and the converse is trivial since $\mathbb{Z} \subseteq \mathbb{Q}$. □

5.4. The Derivation of Exponents $([e_{x_i}])_{i=1}^{\ell}$

Take any product $[q_{x/x'}]$. Then let $\succeq$ be the ordering defined by $x \succ x'$ iff $q_{x/x'} = \infty$. The well-definition of $\succeq$ follows from the fact that any product is a dispersion (by Remark 3.2) and from the fact that a dispersion over X is equivalent to [1] the ordering $\succeq$ and [2] a full-support probability distribution within each equivalence class of $\succeq$ (by McLennan (1989b, page 147) as reformulated by Streufert (2005, sentence containing note 4).

The following lemma uses the cancellation laws in the definition of producthood to derive an additive representation for the ordering $\succeq$. Although the lemma's proof relies on Lemma 5.3 alone, the idea of using cancellation laws to derive an additive representation for an ordering is due to Scott (1964). Further, Krantz, Luce, Suppes, and Tversky (1971, Subsection 9.2) place Scott's insight within a broader context.

Lemma 5.4. *Suppose $[q_{x/x'}]$ is a product over $(X_i)_{i=1}^{\ell}$. Then there exist integers $([e_{x_i}])_{i=1}^{\ell}$ such that $(\forall x, x')$ $x \succeq x'$ iff $\Sigma_{i=1}^{\ell} e_{x_i} \geq \Sigma_{i=1}^{\ell} e_{x'_i}$.*

Proof. We begin with three simple observations about $\succeq$. First, as with any dispersion, reciprocity (8) yields that the ordering $\succeq$ satisfies

$$q_{x/x'} = \infty \text{ iff } x \succ x'$$
$$q_{x/x'} \in (0,\infty) \text{ iff } x \approx x'$$
$$q_{x/x'} = 0 \text{ iff } x \prec x'$$

for any x and x'. Second, the set $\approx$ is nonempty since it must contain all x/x' for which $x = x'$. And third, we may assume that the set $\succ$ is nonempty, for if $\succ$ were empty, this lemma's conclusion could be immediately derived by setting $([e_{x_i}])_{i=1}^{\ell}$ to zero.

Notation. The following five paragraphs will construct a large matrix equation. The notation is daunting.

To begin, note that for any x, we can define the giant row vector $1_x \in \{0,1\}^{\cup_{i=1}^{\ell} X_i}$ in the following fashion: first fix $x = x_1 x_2 ... x_n$, second construct the unit row vector $1_{x_i} \in \{0,1\}^{X_i}$ in each dimension i, and third concatenate these row vectors across the ℓ dimensions to arrive at the giant row vector $1_x = 1_{x_1} 1_{x_2} ... 1_{x_\ell} \in \{0,1\}^{\cup_{i=1}^{\ell} X_i}$. For example, if $X_1 = \{F, G, H\}$ and $X_2 = \{f, g\}$, then $1_{Ff} = 1_F 1_f = [1\ 0\ 0\ 1\ 0]$ because $1_F = [1\ 0\ 0]$ and $1_f = [1\ 0]$.

We will now construct a matrix P whose rows are indexed by the elements of $\succ$ and whose columns are indexed by the elements of $\bigcup_{i=1}^{\ell} X_i$. This matrix P is defined by stating that the row indexed by $x/x' \in \succ$ is $1_x - 1_{x'} \in \{0,1\}^{\cup_{i=1}^{\ell} X_i}$. For example, if $Ff/Fg \in \succ$ then the row of P indexed by Ff/Fg equals $1_{Ff} - 1_{Fg}$, which equals $[1\ 0\ 0\ 1\ 0] - [1\ 0\ 0\ 0\ 1]$, which works out to $[0\ 0\ 0\ 1\ {-1}]$.

Similarly, we will construct a matrix A whose rows are indexed by the elements of $\approx$. This matrix A is defined by stating that the row indexed by $x/x' \in \approx$ is $1_x - 1_{x'} \in \{0,1\}^{\cup_{i=1}^{\ell} X_i}$. For example, if Fg/Fg is an element of $\approx$ (which must be the case), then the row of A indexed by Fg/Fg equals $[0\ 0\ 0\ 0\ 0]$.

Now consider multiplying each row of P by a nonnegative integer and then adding up all these multiplied rows. That is, consider $\pi \in \mathbb{Z}_+^{\succ}$ and $\pi^T P \in \mathbb{Z}^{\cup_{i=1}^{\ell} X_i}$ (here π^T denotes the transpose of the column vector π). To make sure you're with me, $\pi_{x/x'} = 14$ means implicitly that x/x' is an element of $\succ$ (else there is would be no row of P indexed by x/x') and means explicitly that the sum $\pi^T P$ includes 14 copies of the row of P indexed by x/x'.

Similarly, consider multiplying each row of A by an integer and then adding up all its multiplied rows. That is, consider $\alpha \in \mathbb{Z}^{\approx}$ and $\alpha^T A \in \mathbb{Z}^{\bigcup_{i=1}^{\ell} X_i}$. Note that the multipliers in α can be negative (in contrast to the multipliers in π, which cannot be negative).

Apply Producthood. This and the next four paragraphs will use the producthood of $[q_{x/x'}]$ to show that the rows of P and A are "independent" in the sense of (22). In particular, it will be shown that there cannot be a $\pi \in \mathbb{Z}_+^{\succ} \sim \{0\}$ and an $\alpha \in \mathbb{Z}^{\approx}$ such that $\pi^T P + \alpha^T A = 0$. In order to prove this by contradiction, suppose there were such a π and α.

To address the unfortunate fact that α might contain negative elements, define the vector $\hat{\alpha} \in \mathbb{Z}_+^{\approx}$ by

$$\hat{\alpha}_{x/x'} = \begin{pmatrix} \alpha_{x/x'} - \alpha_{x'/x} & \text{if } \alpha_{x/x'} \geq \alpha_{x'/x} \\ 0 & \text{if } \alpha_{x/x'} < \alpha_{x'/x} \end{pmatrix} . \tag{28}$$

We will see that

$$\begin{aligned} \alpha^T A = \; & \Sigma_{x/x' \in \approx} \alpha_{x/x'} (1_x - 1_{x'}) \\ = \; & \Sigma_{x/x' \in (=)} \alpha_{x/x'} (1_x - 1_{x'}) + \Sigma_{x/x' \in (\approx \sim =)} \alpha_{x/x'} (1_x - 1_{x'}) \\ = \; & \Sigma_{x/x' \in (=)} \hat{\alpha}_{x/x'} (1_x - 1_{x'}) + \Sigma_{x/x' \in (\approx \sim =)} \hat{\alpha}_{x/x'} (1_x - 1_{x'}) \\ = \; & \Sigma_{x/x' \in \approx} \hat{\alpha}_{x/x'} (1_x - 1_{x'}) \\ = \; & \hat{\alpha}^T A \; . \end{aligned} \tag{29}$$

Consider the third equality (the others are obvious). Here the first sum is zero regardless of the multipliers because $x = x'$ yields that $1_x - 1_{x'}$ is the zero vector. Further, the indices in the second sum come in pairs because $\approx$ is an equivalence relation, and the sum over any such pair is

$$\begin{aligned} & \alpha_{x/x'} (1_x - 1_{x'}) + \alpha_{x'/x} (1_{x'} - 1_x) \\ =_1 \; & \alpha_{x/x'} (1_x - 1_{x'}) - \alpha_{x'/x} (1_x - 1_{x'}) \\ =_2 \; & (\alpha_{x/x'} - \alpha_{x'/x})(1_x - 1_{x'}) \\ =_3 \; & (\hat{\alpha}_{x/x'} - \hat{\alpha}_{x'/x})(1_x - 1_{x'}) \\ =_4 \; & \hat{\alpha}_{x/x'} (1_x - 1_{x'}) - \hat{\alpha}_{x'/x} (1_x - 1_{x'}) \\ =_5 \; & \hat{\alpha}_{x/x'} (1_x - 1_{x'}) + \hat{\alpha}_{x'/x} (1_{x'} - 1_x) \; , \end{aligned}$$

where $=_3$ holds by definition (28). Hence $\alpha^T A = \hat{\alpha}^T A$, and consequently, $\pi^T P + \hat{\alpha}^T A = 0$. The next three paragraphs will argue that this equality leads to a contradiction.

We now construct an indexed set of elements from $\succ$ and $\approx$. This indexed set will be denoted $\{x^k/x'^k\}_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1}$ (here $\pi\cdot 1$ is the dot product yielding the sum of all the multipliers for elements of $\succ$ and $\hat{\alpha}\cdot 1$ is the sum of all the multipliers for elements of $\approx$). Construct this indexed set so that each element x/x' of $\succ$ appears $\pi_{x/x'}$ times in the first $\pi\cdot 1$ elements of the set, and so that each element x/x' of $\approx$ appears $\hat{\alpha}_{x/x'}$ times in the remaining $\hat{\alpha}\cdot 1$ elements of the set.

This paragraph's conclusion will contradict the next paragraph's conclusion. Note that for each $k \in \{1, 2, ...\pi\cdot 1\}$, we have that $x^k/x'^k \in \succ$ and hence that $q_{x^k/x'^k} = \infty$. Further, there must be at least one such i because $\pi \neq 0$ by assumption (in the first paragraph under "Applying Producthood"). Similarly, for each $k \in \{\pi\cdot 1{+}1, \pi\cdot 1{+}2, \ldots\ \pi\cdot 1{+}\hat{\alpha}\cdot 1\}$, we have that $x^k/x'^k \in \approx$ and hence that $q_{x^k/x'^k} \in (0, \infty)$. Hence, it must be the case that $\odot_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1} q_{x^k/x'^k} = \{\infty\}$.

On the other hand, there might be some potential for cancellation within the expression $\odot_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1} q_{x^k/x'^k}$. In fact, there's a lot of it. The construction of the indexed set $\{x^k/x'^k\}_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1}$ two paragraphs ago yields that

$$\Sigma_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1}(1_{x^k} - 1_{x'^k}) = \pi^T P + \hat{\alpha}^T A\ .$$

Hence, the equality $\pi^T P + \hat{\alpha}^T A = 0$ is equivalent to

$$\Sigma_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1}(1_{x^k} - 1_{x'^k}) = 0\ . \tag{30}$$

This is a row-vector equation in $|\bigcup_{i=1}^{\ell} X_i|$ dimensions, and each of its columns corresponds to some x_i in some dimension i. Hence, the equation is equivalent to

$$(\forall i)(\forall x_i)\ \Sigma_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1}\ 1(x_i{=}x_i^k) - 1(x_i{=}x_i'^k) = 0\ ,$$

where $1(\cdot)$ is the indicator function assuming a value of 1 if its argument is true and 0 if its argument is false. Consider any i and x_i. The term $1(x_i{=}x_i^k)$ is 1 precisely when x_i appears in the numerator of x^k/x'^k, and the term $1(x_i{=}x_i'^k)$ is 1 precisely when x_i appears in the denominator of x^k/x'^k. Accordingly, the above equation holds at x_i precisely when x_i appears in the numerators of $\{x^k/x'^k\}_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1}$ as often as it appears in the denominators of $\{x^k/x'^k\}_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1}$. Since this holds for all x_i, there must be a permutation σ_i such that $(\forall k)\ x_i^k = (x')_i^{\sigma_i(k)}$. Further, since this holds for all i, there exists a permutation vector σ such that $(\forall k)\ x^k = (x')^{\sigma,k}$. This equality and producthood yield

$$\odot_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1} q_{x^k/x'^k} = \odot_{k=1}^{\pi\cdot 1+\hat{\alpha}\cdot 1} q_{(x')^{\sigma,k}/x'^k} \ni 1. \tag{31}$$

This contradicts the conclusion of the previous paragraph.

Applying Lemma 5.3. Now consider Lemma 5.3 applied at this P and A (note that its p is $|\succ|$, its a is $|\approx|$, and its k is $|\bigcup_{i=1}^{\ell} X_i|$). The preceding five paragraphs have shown that the rows of P and A are "independent" in the sense of (22). Hence Lemma 5.3 yields the existence of some $w \in \mathbb{Z}^{|\cup_{i=1}^{\ell} X_i|}$ such that $Pw \gg 0$ and $Aw = 0$. Define the exponents $([e_{x_i}])_{i=1}^{\ell}$ by setting

$$[e_{x_1}][e_{x_2}]...[e_{x_\ell}] = w \ ,$$

where $[e_{x_1}][e_{x_2}]...[e_{x_\ell}]$ is the giant column vector obtained by concatenating the vectors in $([e_{x_i}])_{i=1}^{\ell}$.

If $x'' \succ x'$, then the vector inequality $Pw \gg 0$ at the row indexed by $x''/x' \in \succ$ is equivalent to

$$(1_{x''}-1_{x'})([e_{x_1}][e_{x_2}]...[e_{x_\ell}]) > 0 \ ,$$

which is equivalent to $\Sigma_{i=1}^{\ell} e_{x_i''} > \Sigma_{i=1}^{\ell} e_{x_i'}$. Similarly, if $x'' \approx x'$, then the vector equality $Aw = 0$ at the row indexed by $x''/x' \in \approx$ is equivalent to

$$(1_{x''}-1_{x'})([e_{x_1}][e_{x_2}]...[e_{x_\ell}]) = 0 \ ,$$

which is equivalent to $\Sigma_{i=1}^{\ell} e_{x_i''} = \Sigma_{i=1}^{\ell} e_{x_i'}$. In summary, $x'' \succ x'$ implies $\Sigma_{i=1}^{\ell} e_{x_i''} > \Sigma_{i=1}^{\ell} e_{x_i'}$, and $x'' \approx x'$ implies $\Sigma_{i=1}^{\ell} e_{x_i''} = \Sigma_{i=1}^{\ell} e_{x_i'}$.

These two facts directly yield that $x \succeq x'$ implies $\Sigma_{i=1}^{\ell} e_{x_i} \geq \Sigma_{i=1}^{\ell} e_{x_i'}$. Conversely, *not* $x \succeq x'$ implies $x \prec x'$ by the completeness of $\succeq$, which implies $\Sigma_{i=1}^{\ell} e_{x_i} < \Sigma_{i=1}^{\ell} e_{x_i'}$ by the first of the last paragraph's facts, which implies *not* $\Sigma_{i=1}^{\ell} e_{x_i} \geq \Sigma_{i=1}^{\ell} e_{x_i'}$. Therefore, $x \succeq x'$ if and only if $\Sigma_{i=1}^{\ell} e_{x_i} \geq \Sigma_{i=1}^{\ell} e_{x_i'}$. □

5.5. Deriving Coefficients $([c_{x_i}])_{i=1}^{\ell}$.

Theorem 5.1($b \Rightarrow a$) can now be proven by citing Lemma 5.4 to obtain exponents, and then working to derive the coefficients.

Proof 5.5 (for Theorem 5.1($b \Rightarrow a$)). Take any product $[q_{x/x'}]$. Our task is to show that $[q_{x/x'}]$ is represented by some $[\Pi_{i=1}^{\ell} c_{x_i} n^{e_{x_i}}]$. By the observation at (16), this means that we are to find $([e_{x_i}])_{i=1}^{\ell}$ and $([c_{x_i}])_{i=1}^{\ell}$ such that

$$(\forall x, x') \ q_{x/x'} = \begin{pmatrix} \infty & \text{if } \Sigma_{i=1}^{\ell} e_{x_i} > \Sigma_{i=1}^{\ell} e_{x_i'} \\ (\Pi_{i=1}^{\ell} c_{x_i})/(\Pi_{i=1}^{\ell} c_{x_i'}) & \text{if } \Sigma_{i=1}^{\ell} e_{x_i} = \Sigma_{i=1}^{\ell} e_{x_i'} \\ 0 & \text{if } \Sigma_{i=1}^{\ell} e_{x_i} < \Sigma_{i=1}^{\ell} e_{x_i'} \end{pmatrix} \ .$$

By Lemma 5.4, and by the definition of $\succeq$ at the start of Subsection 5.4, we have $([e_{x_i}])_{i=1}^{\ell}$ such that

(32a) $$q_{x/x'} = \infty \text{ iff } x \succ x' \text{ iff } \Sigma_{i=1}^{\ell} e_{x_i} > \Sigma_{i=1}^{\ell} e_{x'_i}$$

(32b) $$q_{x/x'} \in (0,\infty) \text{ iff } x \approx x' \text{ iff } \Sigma_{i=1}^{\ell} e_{x_i} = \Sigma_{i=1}^{\ell} e_{x'_i}$$

(32c) $$q_{x/x'} = 0 \text{ iff } x \prec x' \text{ iff } \Sigma_{i=1}^{\ell} e_{x_i} < \Sigma_{i=1}^{\ell} e_{x'_i} \ .$$

Thus it remains to find (positive) coefficients $([c_{x_i}])_{i=1}^{\ell}$ such that

$$(\forall x/x' {\in} {\approx})\ (\Pi_{i=1}^{\ell} c_{x_i})/(\Pi_{i=1}^{\ell} c_{x'_i}) = q_{x/x'} \ .$$

Since (32b) yields the critical fact that $q_{x/x'} \in (0,\infty)$ for every x/x' in $\approx$, this is equivalent to finding real numbers $([d_{x_i}])_{i=1}^{\ell}$ such that

$$(\forall x/x' {\in} {\approx})\ \Sigma_{i=1}^{\ell} d_{x_i} - \Sigma_{i=1}^{\ell} d_{x'_i} = \ln(q_{x/x'}) \ .$$

Applying Linear Algebra. As in the proof of Lemma 5.4 in the previous subsection, we can define for any $x = x_1 x_2 ... x_\ell$ the giant row vector $1_x = 1_{x_1} 1_{x_2} ... 1_{x_\ell} \in \{0,1\}^{\cup_{i=1}^{\ell} X_i}$ by concatenating the unit vectors $(1_{x_i})_{i=1}^{\ell}$ across i. Using this notation, the system becomes

$$(\forall x/x' {\in} {\approx})\ (1_x {-} 1_{x'}) d = \ln(q_{x/x'}) \ ,$$

where d is the giant column vector $d = [d_{x_1}][d_{x_2}]...[d_{x_\ell}]$ obtained by concatenating together the variables $([d_{x_i}])_{i=1}^{\ell}$. Notice that this is a matrix equation of the form $Ad = b$, in which row x/x' of the coefficient matrix A is $1_x {-} 1_{x'}$ and element x/x' in the vector b is $\ln(q_{x/x'})$.

Recall from elementary linear algebra that Gaussian elimination is equivalent to premultiplying the augmented matrix $[A\,b]$ with a certain square matrix E which replicates the elementary row operations and row permutations. Further recall that back substitution then reveals a solution to $Ad = b$ provided that $E[A\,b]$ does not contain a row which is zero in all but the last column (see for example Strang (1980, Chapter 1)). In the present circumstance, E has only rational elements because the coefficient matrix A has only rational elements. As a result, each row in $E[A\,b]$ can be written as

$$[\ \Sigma_{x/x' \in \approx} a_{x/x'} (1_x {-} 1_{x'}) \ \ \Sigma_{x/x' \in \approx} a_{x/x'} \ln(q_{x/x'})\ ]$$

for some rational vector $(a_{x/x'})_{x/x' \in \approx}$ equal to a row of E. Thus $E[A\,b]$ does not have a row in which all but the last column is zero if

(33) $$\Sigma_{x/x' \in \approx} a_{x/x'} (1_x {-} 1_{x'}) = 0 \ \text{ implies } \ \Sigma_{x/x' \in \approx} a_{x/x'} \ln(q_{x/x'}) = 0$$

for all rational vectors $(a_{x/x'})_{x/x' \in \approx}$. We will establish this conditional to complete the proof.

To set the argument up, note that statement (33) holds if

$$(34)\qquad \Sigma_{x/x'\in\approx}\alpha_{x/x'}(1_x{-}1_{x'}) = 0 \;\text{ implies }\; \Sigma_{x/x'\in\approx}\alpha_{x/x'}\ln(q_{x/x'}) = 0$$

for all *integer* vectors $(\alpha_{x/x'})_{x/x'\in\approx}$ (to see the contrapositive of this claim, note that if $(a_{x/x'})_{x/x'\in\approx}$ violates (33) then some multiple of $(a_{x/x'})_{x/x'\in\approx}$ containing only integers violates (34)). Accordingly, assume the row-vector equation

$$(35)\qquad \Sigma_{x/x'\in\approx}\alpha_{x/x'}(1_x{-}1_{x'}) = 0$$

for some integer vector $(\alpha_{x/x'})_{x/x'\in\approx}$. The remainder of this proof will then establish the scalar equation

$$(36)\qquad \Sigma_{x/x'\in\approx}\alpha_{x/x'}\ln(q_{x/x'}) = 0\ .$$

Applying Producthood. The assumption (35) is the same as $\alpha^T A = 0$, where α is the integer vector $(\alpha_{x/x'})_{x/x'\in\approx}$ and A is the matrix whose rows are indexed by the elements of $\approx$ and whose row at x/x' is $1_x{-}1_{x'}$ (as under "Notation" in the last subsection). To address the unfortunate fact that some elements of α may be negative, define $\hat{\alpha}$ by

$$(37)\qquad \hat{\alpha}_{x/x'} = \begin{pmatrix} \alpha_{x/x'} - \alpha_{x'/x} & \text{if } \alpha_{x/x'} \geq \alpha_{x'/x} \\ \alpha_{x'/x} - \alpha_{x/x'} & \text{if } \alpha_{x/x'} < \alpha_{x/x'} \end{pmatrix}\ .$$

As at (29), $\hat{\alpha}^T A = \alpha^T A$, and hence the assumption $\alpha^T A = 0$ implies $\hat{\alpha}^T A = 0$.

Further, as in the paragraph after (29), we can construct an indexed set $\{x^k/x'^k\}_{k=1}^{\hat{\alpha}\cdot 1}$ assigning elements of $\approx$ so that each element x^k/x'^k of $\approx$ is assigned exactly $\hat{\alpha}_{x/x'}$ times. The assumption $\hat{\alpha}^T A = 0$ is then equivalent to $\Sigma_{k=1}^{\hat{\alpha}\cdot 1}(1_{x^k}{-}1_{x'^k}) = 0$. As in the seven sentences after (30), this implies the existence of a permutation vector σ such that $(\forall k)\ x^k = (x')^{\sigma,k}$. This and producthood yield that

$$\odot_{k=1}^{\hat{\alpha}\cdot 1} q_{x^k/x'^k} = \odot_{k=1}^{\hat{\alpha}\cdot 1} q_{(x')^{\sigma,k}/x'^k} \ni 1\ .$$

By (32b), this is equivalent to

$$\Pi_{k=1}^{\hat{\alpha}\cdot 1} q_{x^k/x'^k} = 1\ ,$$

which is equivalent to

$$\Sigma_{k=1}^{\hat{\alpha}\cdot 1}\ln(q_{x^k/x'^k}) = 0\ ,$$

which by the definition of $(x^k/x'^k)_{k=1}^{\hat{\alpha}\cdot 1}$ is equivalent to

$$\Sigma_{x/x'\in\approx}\hat{\alpha}_{x/x'}\ln(q_{x/x'}) = 0\ .$$

It only remains to show that the above equation is equivalent to (36). This paragraph accomplishes that by deriving

$$\begin{aligned}
&\Sigma_{x/x' \in \approx} \hat{\alpha}_{x/x'} \ln(q_{x/x'}) \\
&= \Sigma_{x/x' \in (=)} \hat{\alpha}_{x/x'} \ln(q_{x/x'}) + \Sigma_{x/x' \in (\approx \sim =)} \hat{\alpha}_{x/x'} \ln(q_{x/x'}) \\
&= \Sigma_{x/x' \in (=)} \alpha_{x/x'} \ln(q_{x/x'}) + \Sigma_{x/x' \in (\approx \sim =)} \alpha_{x/x'} \ln(q_{x/x'}) \\
&= \Sigma_{x/x' \in \approx} \alpha_{x/x'} \ln(q_{x/x'}) \ .
\end{aligned}$$

Consider the second equality (the others are obvious). Here the first sum is zero because $x = x'$ yields $\ln(q_{x/x'}) = \ln(1) = 0$ by unit diagonality (6). Further, the indices in the second sum come in pairs because $\approx$ is an equivalence relation and the sum over any such pair is

$$\begin{aligned}
&\hat{\alpha}_{x/x'} \ln(q_{x/x'}) + \hat{\alpha}_{x'/x} \ln(q_{x'/x}) \\
=_1\ &\hat{\alpha}_{x/x'} \ln(q_{x/x'}) - \hat{\alpha}_{x'/x} \ln(q_{x/x'}) \\
=_2\ &(\hat{\alpha}_{x/x'} - \hat{\alpha}_{x'/x}) \ln(q_{x/x'}) \\
=_3\ &(\alpha_{x/x'} - \alpha_{x'/x}) \ln(q_{x/x'}) \\
=_4\ &\alpha_{x/x'} \ln(q_{x/x'}) - \alpha_{x'/x} \ln(q_{x/x'}) \\
=_5\ &\alpha_{x/x'} \ln(q_{x/x'}) + \alpha_{x'/x} \ln(q_{x'/x}) \ ,
\end{aligned}$$

where $=_3$ holds by (37) and $=_1$ and $=_5$ hold by reciprocity (8). □

5.6. Representation of Marginals

Proof 5.6 (of Theorem 5.1's sentence about marginals). Let $[q_{x/x'}]$ be the product represented by $[\Pi_{i=1}^{\ell} c_{x_i} n^{e_{x_i}}]$. Fix any $x^\star$, and consider any i.

First, by Remark 4.1, the marginal with respect to x_i is $[q_{x_i x^\star_{-i}/x'_i x^\star_{-i}}]$. Second, since $[q_{x_i x^\star_{-i}/x'_i x^\star_{-i}}]$ is a restriction of $[q_{x/x'}]$ and since all of $[q_{x/x'}]$ is represented by $[\Pi_{k=1}^{\ell} c_{x_k} n^{e_{x_k}}]$, we have that $[q_{x_i x^\star_{-i}/x'_i x^\star_{-i}}]$ is represented by $[c_{x_i} n^{e_{x_i}}](\Pi_{k \neq i} c_{x^\star_k} n^{e_{x^\star_k}})$. The last two sentences together yield that the marginal with respect to x_i is represented by $[c_{x_i} n^{e_{x_i}}](\Pi_{k \neq i} c_{x^\star_k} n^{e_{x^\star_k}})$.

Note that $(\Pi_{k \neq i} c_{x^\star_k} n^{e_{x^\star_k}})$ is constant with respect to x_i. Also note that the definition (15) of representation depends only on the ratio between monomials. The last three sentences together yield that the marginal with respect to x_i is represented by $[c_{x_i} n^{e_{x_i}}]$. □

6. A Little Topology

6.1. $\Delta(X_i)_{i=1}^{\ell}$ is Compact

Up until now, the paper has been purely algebraic. Now put the usual topology on $[0,\infty]$ and the corresponding product topology on any (finite) product of $[0,\infty]$ with itself.

Theorem 6.1. *Take any $(X_i)_{i=1}^{\ell}$ and let $X{=}\Pi_{i=1}^{\ell}X_i$. Then $\Delta(X_i)_{i=1}^{\ell}$ is a compact subset of $[0,\infty]^{X^2}$.*

The theorem's proof comes after the following lemma.

Lemma 6.2. *Let $\langle(u_j^n)_{j=0}^m\rangle_n$ be a sequence of vectors in $[0,\infty]^{1+m}$ which converges to the vector $(u_j^\star)_{j=0}^m$. If each vector in the sequence satisfies $1 \in \odot(u_j^n)_{j=0}^m$ then the limit vector satisfies $1 \in \odot(u_j^\star)_{j=0}^m$.*

Proof. Take any sequence $\langle(u_j^n)_{j=0}^m\rangle_n$ converging to $(u_j^\star)_{j=0}^m$. We will derive the conditional in the lemma's second sentence in each of four cases:

Case	$(\exists j)u_j^\star{=}0$	$(\exists j)u_j^\star{=}\infty$
1	true	true
2	false	true
3	true	false
4	false	false

Case 1. Throughout this case, the conditional's conclusion is true because $\odot(u_j^\star)_{j=0}^m = [0,\infty]$.

Case 2. Throughout this case the conditional's assumption is false. Specifically, since all of the dimensions j converge to a positive number, there is some positive v and some index n^0 such that

$$(\forall n{>}n^0)(\forall j)\ u_j^n > v\ . \tag{38}$$

Then let j^∞ denote a dimension in which $u_{j^\infty}^\star = \infty$ and note that there exists some index $n^\infty > n^0$ such that

$$(\forall n{>}n^\infty)\ u_{j^\infty}^n > (1/v)^m\ . \tag{39}$$

Together (38) and (39) yield

$$(\forall n{>}n^\infty)\ \Pi_{j=0}^m u_j^n = u_{j^\infty}^n{\cdot}\Pi_{j\neq j^\infty}u_j^n > (1/v)^m v^m = 1$$

(all products are well-defined since all terms are positive). In casual terms, the product of a vector in the sequence is ultimately greater than one. Thus, it is cannot the case that all vectors in the sequence satisfy $1 \in \odot(u_j^n)_{j=0}^m$ (in fact, no more than a finite number of them can).

Case 3. This case is symmetric to Case 2: the conditional's assumption is always false because the product of a vector in the sequence is ultimately less than one.

Case 4. Since $\langle(u_j^n)_{j=0}^m\rangle_n$ converges to a finite vector, there is some n^0 after which every vector u_j^n is in ordinary Euclidean space $\mathbb{R}^{1+m}$. The lemma's conditional then follows from the algebra of limits (that is, from Rudin (1976) Theorem 3.3(c)). In particular, the conditional's assumption implies $(\forall n{>}n^0)\ 1 = \Pi_{j=0}^m u_j^n$ which implies $1 = \Pi_{j=0}^m u_j^\star$ which implies the conditional's conclusion. □

Proof of Theorem 6.1. Since $[0,\infty]^{X^2}$ is compact, we need only show that $\Delta(X_i)_{i=1}^\ell$ is closed. Accordingly, take any sequence $\langle[q_{x/x'}^n]\rangle_n$ in $\Delta(X_i)_{i=1}^\ell$ which converges to some $[q_{x/x'}^\star]$. Our task is to show that $[q_{x/x'}^\star]$ is in $\Delta(X_i)_{i=1}^\ell$. In other words, by definition (4), our task is to show

$$(\forall m)(\forall\sigma)(\forall(x^j)_{j=0}^m)\ 1 \in \odot(q^\star_{x^{\sigma,j}/x^j})_{j=0}^m\ .$$

Accordingly, fix any order m, any permutation vector σ, and any instance $(x^j)_{j=0}^m$. Our task is to show

$$1 \in \odot(q^\star_{x^{\sigma,j}/x^j})_{j=0}^m\ . \tag{40}$$

Since each $[q_{x/x'}^n]$ is in $\Delta(X_i)_{i=1}^\ell$ by assumption, the definition (4) gives us that

$$1 \in \odot(q^n_{x^{\sigma,j}/x^j})_{j=0}^m\ .$$

Hence Lemma 6.2 applied at $\langle(u_j^n)_{j=0}^m\rangle_n = \langle(q^n_{x^{\sigma,j}/x^j})_{j=0}^m\rangle_n$ and $(u_j^\star)_{j=0}^m = q^\star_{x^{\sigma,j}/x^j}$ yields (40). □

6.2. Positive Products

A *positive product* is an element of the set $\Delta^\circ(X_i)_{i=1}^\ell$ defined by

$$\{\ [q_{x/x'}] \in \Delta(X_i)_{i=1}^\ell \mid (\forall x,x')\ q_{x/x'} \in (0,\infty)\ \}\ . \tag{41}$$

By reciprocity (8), the restriction $(\forall x,x')\ q_{x/x'} \in (0,\infty)$ is equivalent to both $(\forall x,x')\ q_{x/x'} \in (0,\infty]$ and $(\forall x,x')\ q_{x/x'} \in [0,\infty)$. Accordingly, "positive products," "finite products," and "positive finite products" are all synonymous.

Remark 6.3. *$\Delta(X_i)_{i=1}^\ell$ is the closure of $\Delta^\circ(X_i)_{i=1}^\ell$.*

Proof. ⊇. $\Delta(X_i)_{i=1}^\ell$ contains the closure of $\Delta^\circ(X_i)_{i=1}^\ell$ since $\Delta(X_i)_{i=1}^\ell$ contains $\Delta^\circ(X_i)_{i=1}^\ell$ by definition (41) and since $\Delta(X_i)_{i=1}^\ell$ is compact by Theorem 6.1.

$\subseteq$. Take any $[q_{x/x'}]$ in $\Delta(X_i)_{i=1}^{\ell}$. By Theorem 5.1($b \Rightarrow a$) and the observation at (16), there exist $([c_{x_i}])_{i=1}^{\ell}$ and $([e_{x_i}])_{i=1}^{\ell}$ such that

$$(\forall x,x')\ q_{x/x'} = \begin{pmatrix} \infty & \text{if } \Sigma_{n=1}^{\ell} e_{x_i} > \Sigma_{n=1}^{\ell} e_{x'_i} \\ (\Pi_{n=1}^{\ell} c_{x_i})/(\Pi_{n=1}^{\ell} c_{x'_i}) & \text{if } \Sigma_{n=1}^{\ell} e_{x_i} = \Sigma_{n=1}^{\ell} e_{x'_i} \\ 0 & \text{if } \Sigma_{n=1}^{\ell} e_{x_i} < \Sigma_{n=1}^{\ell} e_{x'_i} \end{pmatrix} .$$

Hence

$$(\forall x,x')\ q_{x/x'} = \lim_{n \to \infty} \frac{(\Pi_{n=1}^{\ell} c_{x_i}) n^{\Sigma_{n=1}^{\ell} e_{x_i}}}{(\Pi_{n=1}^{\ell} c_{x'_i}) n^{\Sigma_{n=1}^{\ell} e_{x'_i}}} . \tag{42}$$

Now define $\langle [q^n_{x/x'}] \rangle_n$ by

$$(\forall n)(\forall x,x')\ q^n_{x/x'} = \frac{\Pi_{n=1}^{\ell} c_{x_i} n^{e_{x_i}}}{\Pi_{n=1}^{\ell} c_{x'_i} n^{e_{x'_i}}} .$$

First note that $\langle [q^n_{x/x'}] \rangle_n$ converges to $[q_{x/x'}]$ by (42). Second note that every $[q^n_{x/x'}]$ is in $\Delta^{\circ}(X_i)_{i=1}^{\ell}$ because every relative probability $q^n_{x/x'}$ is positive and because all the cancellation laws in the definition (4) of $\Delta(X_i)_{i=1}^{\ell}$ are satisfied by ordinary real algebra. These two observations yield that $[q_{x/x'}]$ is in the closure of $\Delta^{\circ}(X_i)_{i=1}^{\ell}$. □

Remark 6.3 is equivalent to a reformulation of Theorem 2.10 in Kohlberg and Reny (1997), which shows that a type of acyclicity is equivalent to their concept of strong independence. Specificially, their acyclicity is equivalent to producthood by Streufert (2003, Remark B.6($a \Leftrightarrow a^{KR}$)), and their strong independence is equivalent to membership in the closure of the set of positive products because there is a one-to-one correspondence between the set of positive products and the set of ordinary, full-support, product distributions. This result of Kohlberg and Reny (1997) appears to be the closest predecessor of any of this paper's results.

Appendix A. Δ is Not an Iterated Binary Operation

A.1. Overview

This appendix demonstrates that ℓ-dimensional producthood is more restrictive than $\ell-1$ iterative applications of 2-dimensional producthood. In this sense, ℓ-dimensional producthood is more restrictive than one might guess.

To make these observations concrete, we require temporary notation for a binary operation $\otimes$. There are two steps. First, let $[q_{y/y'}]\otimes[q_{z/z'}]$ denote the set of all products of some table $[q_{y/y'}]$ over some set Y with some table $[q_{z/z'}]$ over some set Z. In other words, let $[q_{y/y'}]\otimes[q_{z/z'}]$ be the set of all products over (Y, Z) that have marginals $[q_{y/y'}]$ and $[q_{z/z'}]$. To exercise this first definition, note that $[q_{y/y'}]\otimes[q_{z/z'}]$ typically has many elements (because marginals are ambiguous), and that $[q_{y/y'}]\otimes[q_{z/z'}]$ is empty if either $[q_{x/x'}]$ or $[q_{y/y'}]$ is not a dispersion (because marginals are dispersions by definition).

Second, for any set Q_Y of tables $[q_{y/y'}]$ over Y, and for any set Q_Z of tables $[q_{z/z'}]$ over Z, let $Q_Y\otimes Q_Z$ be the set of all products of some table in Q_Y with some table in Q_Z. To exercise this second definition, note that

$$(\forall X_1, X_2)\ \Delta(X_1, X_2) = \Delta(X_1)\otimes\Delta(X_2) \tag{43}$$

because marginals are dispersions by definition.

Surprisingly, Remark A.1 shows that

$$(\forall X_1, X_2, X_3)\ \Delta(X_1, X_2, X_3) = \Delta(X_1)\otimes(\Delta(X_2)\otimes\Delta(X_3))$$

is *false*. Rather, $\Delta(X_1, X_2, X_3)$ is a subset, and typically a strict subset, of $\Delta(X_1)\otimes(\Delta(X_2)\otimes\Delta(X_3))$. In this precise sense, producthood over ℓ dimensions is more restrictive than $\ell-1$ iterative applications of producthood over 2 dimensions.

To appreciate this intuitively, recall that an element of $\Delta(X_1, X_2, X_3)$ must satisfy the cancellation laws for all permutation vectors $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Meanwhile, an element of $\Delta(X_1)\otimes(\Delta(X_2)\otimes\Delta(X_3))$ must satisfy this same cancellation condition for certain classes of permutation vectors: the outer $\otimes$ concerns vectors of the form

$$\{\ (\sigma_1, \sigma_2, \sigma_3)\ |\ \sigma_2 = \sigma_3\ \}\ ,$$

and the inner $\otimes$ concerns vectors of the form

$$\{\ (\sigma_1, \sigma_2, \sigma_3)\ |\ \sigma_1 \text{ is the identity function }\}\ .$$

Since the class of all permutation vectors contains these two classes, the set of tables obeying cancellation for all permutation vectors is a subset, and typically a strict subset, of the set of tables obeying cancellation just within the two classes. Accordingly, $\Delta(X_1, X_2, X_3)$ is a subset, and typically a strict subset, of $\Delta(X_1)\otimes(\Delta(X_2)\otimes\Delta(X_3))$.

A.2. Details

Remark A.1.

(*a*) $(\forall X_1, X_2, X_3)\ \Delta(X_1, X_2, X_3) \subseteq \Delta(X_1)\otimes(\Delta(X_2)\otimes\Delta(X_3))$ *and*

(*b*) *not* $(\forall X_1, X_2, X_3)\ \Delta(X_1, X_2, X_3) \supseteq \Delta(X_1)\otimes(\Delta(X_2)\otimes\Delta(X_3))$.

Proof. (a) Take any $[q_{x/x'}]$ in $\Delta(X_1,X_2,X_3)$, and for future use, fix any $x^\star \in X$. First, the definition (4) of $\Delta(X_1,X_2,X_3)$ implies that $[q_{x/x'}]$ satisfies

$$(\forall m)(\forall \sigma)\ \Big(\ \sigma_2{=}\sigma_3 \text{ implies } (\forall (x^j)_{j=0}^m)\ 1 \in \odot(q_{x^{\sigma,j}/x^j})_{j=0}^m\ \Big)\ ,$$

which is equivalent to $[q_{x/x'}] \in \Delta(X_1, X_2{\times}X_3)$. Thus by the definition of $\otimes$ and the second sentence of Remark 4.1, we have that

$$[q_{x/x'}] \in \{[q_{x_1x_2^\star x_3^\star/x_1'x_2^\star x_3^\star}]\}{\otimes}\{[q_{x_1^\star x_2x_3/x_1^\star x_2'x_3'}]\}\ .$$

But this is more than we need: it suffices to remember that

$$[q_{x/x'}] \in \Delta(X_1){\otimes}\{[q_{x_1^\star x_2x_3/x_1^\star x_2'x_3'}]\}\ . \tag{44}$$

Second, the definition (4) of $\Delta(X_1,X_2,X_3)$ also implies that $[q_{x_1^\star x_2x_3/x_1^\star x_2'x_3'}]$ satisfies

$$(\forall m)(\forall \sigma_2,\sigma_3)(\forall (x_2^j,x_3^j)_{j=1}^m)\ 1 \in \odot(q_{x_1^\star x_2^{\sigma_2(j)}x_3^{\sigma_3(j)}/x_1^\star x_2^j x_3^j})_{j=1}^m\ ,$$

which is equivalent to $[q_{x_1^\star x_2x_3/x_1^\star x_2'x_3'}] \in \Delta(X_2,X_3)$, which by (43) is equivalent to

$$[q_{x_1^\star x_2x_3/x_1^\star x_2'x_3'}] \in \Delta(X_2){\otimes}\Delta(X_3)\ . \tag{45}$$

(44) and (45) together yield that $[q_{x/x'}] \in \Delta(X_1){\otimes}(\Delta(X_2){\otimes}\Delta(X_3))$.

(b) The remainder of the proof exhibits an example which falls outside of $\Delta(X_1,X_2,X_3)$ but inside of $\Delta(X_1){\otimes}(\Delta(X_2){\otimes}\Delta(X_3))$. Suppose that $X_1 = \{L,R\}$, $X_2 = \{A,B\}$, and $X_3 = \{a,b\}$, and consider the following table over $X_1{\times}X_2{\times}X_3$.

$[q_{x/x'}] =$

RBb	0	0	0	0	0	0	0	1
RBa	0	0	0	1	0	1	1	∞
RAb	0	0	0	1	0	1	1	∞
RAa	0	∞	∞	∞	1	∞	∞	∞
LBb	0	0	0	1	0	1	1	∞
LBa	0	1	1	∞	0	∞	∞	∞
LAb	0	1	1	∞	0	∞	∞	∞
LAa	1	∞	∞	∞	∞	∞	∞	∞
	LAa	LAb	LBa	LBb	RAa	RAb	RBa	RBb

.

This $[q_{x/x'}]$ does not belong to $\Delta(X_1,X_2,X_3)$. Specifically, if it were an element of $\Delta(X_1,X_2,X_3)$, it would satisfy the following instance of a second-order cancellation law,

$$1 \in \odot\{q_{RAb/LBb}, q_{LAb/RAa}, q_{LBa/LAb}\}$$

(cancel terms to verify that suitable permutations exist). Yet, this product actually works out to

$$\odot\{q_{RAb/LBb}, q_{LAb/RAa}, q_{LBa/LAb}\} = \odot\{1, \infty, 1\} = \{\infty\} \ .$$

On the other hand, note that $[q_{Lx_2x_3/Lx'_2x'_3}]$ (that is, the southwest quadrant of $[q_{x/x'}]$) is represented by the product

b	n^{-1}	1
a	n^{-2}	n^{-1}
	A	B

of two monomial vectors. Hence by Theorem 5.1(a⇒b), $[q_{Lx_2x_3/Lx'_2x'_3}]$ is a product of some dispersion over X_2 and some dispersion over X_3. In brief,

$$[q_{Lx_2x_3/Lx'_2x'_3}] \in \Delta(X_2) \otimes \Delta(X_3) \ . \tag{46}$$

Further, $[q_{x/x'}]$ itself is represented by

R	n^{-3}	n^{-1}	n^{-1}	1
L	n^{-4}	n^{-2}	n^{-2}	n^{-1}
	Aa	Ab	Ba	Bb

,

which is the product of a monomial vector over $X_1 = \{L, R\}$ and a monomial vector which represents $[q_{Lx_2x_3/Lx'_2x'_3}]$. Hence by Theorem 5.1(a⇒b), $[q_{x/x'}]$ is a product of some dispersion over X_1 and $[q_{Lx_2x_3/Lx'_2x'_3}]$. In brief,

$$[q_{x/x'}] \in \Delta(X_1) \otimes \{[q_{Lx_2x_3/Lx'_2x'_3}]\} \ . \tag{47}$$

(46) and (47) together imply that $[q_{x/x'}] \in \Delta(X_1) \otimes (\Delta(X_2) \otimes \Delta(X_3))$. □

Appendix B. Real Exponents

Throughout the paper, the symbol e assumes integer values, and accordingly, Theorem 5.1 characterizes producthood by means of monomials with integer exponents. This appendix notes that this result for integer exponents is stronger than an analogous result for real exponents. In particular, Corollary B.1 follows from Theorem 5.1 and two components of its proof. Here $\dot{e}$ denotes a real number, and accordingly, monomials with real exponents have the form $cn^{\dot{e}}$.

Corollary B.1. *Let* $[q_{x/x'}]$ *be a table over* $\Pi_{i=1}^{\ell} X_i$. *Then* ($\dot{a}$) $[q_{x/x'}]$ *is represented by some* $[\Pi_{i=1}^{\ell} c_{x_i} n^{\dot{e}_{x_i}}]$ *iff (b)* $[q_{x/x'}]$ *is a product over*

$(X_i)_{i=1}^{\ell}$. *Furthermore, the marginals of the product represented by* $[\Pi_{i=1}^{\ell} c_{x_i} n^{\dot{e}_{x_i}}]$ *are represented by* $([c_{x_i} n^{\dot{e}_{x_i}}])_{i=1}^{\ell}$.

Proof. ($\dot{\text{a}}$) implies (b) by Proof 5.2 after replacing (a) with ($\dot{\text{a}}$) and e with $\dot{e}$. The converse holds by Theorem 5.1(b⇒a) together with the obvious fact that (a) implies ($\dot{\text{a}}$). The corollary's second sentence follows from Proof 5.6 after replacing e with $\dot{e}$. □

Corollary B.1 is strictly weaker than Theorem 5.1 to the extent that it derives real but not necessarily integer exponents.

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