Supplementary Material to "Good News and Bad News in Two-Armed Bandits", by Braz Camargo:

The definition of weak MLR order, denoted by $\succeq_r$, is necessary in the proof of Lemma 2. Recall that $sp(\pi)$ denotes the support of $\pi \in \Pi$.

Definition 1. Suppose $\pi, \pi' \in \Pi$. Then $\pi \succeq_r \pi'$ if $(sp(\pi) \backslash sp(\pi')) \geq sp(\pi')$, $sp(\pi) \geq (sp(\pi') \backslash sp(\pi))$, and

$$\pi'(s')\pi(s) \leq \pi'(s)\pi(s')$$

for all $s, s' \in \mathrm{S}$ such that $s' \succeq_{\mathrm{S}} s$.[1]

In what follows, the condition on the supports of two arbitrary elements π and π' of Π necessary for both $\pi \succeq_{\text{tp}} \pi'$ and $\pi \succeq_r \pi'$ is referred to as the support condition.

Lemma 1. *Suppose π is TP_2. Then $y \succeq_{a_i} y'$ if, and only if, $B(y, a_i, \pi) \succeq_{\text{tp}} B(y', a_i, \pi)$, $i = 0, 1$.*

Proof: First observe that the assumption that $\lambda(a_i, s)$ has full support implies that the measures $B(y, a_i, \pi)$ and $B(y', a_i, \pi)$ have, for all $y, y' \in \mathrm{Y}$, a common support. Moreover,

$$B(y, a_i, \pi)(s \vee s')B(y', a_i, \pi)(s \wedge s') \geq B(y, a_i, \pi)(s)B(y', a_i, \pi)(s')$$

if, and only if,

$$f(y|a_i, s \vee s')f(y'|a_i, s \wedge s')\pi(s \vee s')\pi(s \wedge s') \geq f(y|a_i, s)f(y'|a_i, s')\pi(s)\pi(s').$$

The result then follows from the definition of $\succeq_{a_i}$ and the fact that π is TP_2 by assumption. $\square$

Lemma 2. *Suppose $\pi \succeq_{\text{tp}} \pi'$ and either π or π' is TP_2. Then $B(y, a, \pi) \succeq_{\text{tp}} B(y, a, \pi')$ for all $y \in \mathrm{Y}$ and $a \in \mathrm{A}$.*

Proof: Suppose, without loss, that π is TP_2. It is straightforward to see that $B(y, a, \pi)$ and $B(y, a, \pi')$ satisfy the support condition necessary for $B(y, a, \pi) \succeq_{\text{tp}} B(y, a, \pi')$. The next step consists in showing that $B(y, a, \pi)$ is TP_2 for all $y \in \mathrm{Y}$ and $a \in \mathrm{A}$. For this, let $s, s' \in \Pi$, $a \in \mathrm{A}$, and $y \in \mathrm{Y}$. Then

$$B(y, a, \pi)(s \vee s')B(y, a, \pi)(s \wedge s') \geq B(y, a, \pi)(s)B(y, a, \pi)(s')$$

if, and only if,

$$f(y|a, s \vee s')f(y|a, s \wedge s')\pi(s \vee s')\pi(s \wedge s') \geq f(y|a, s)f(y|a, s')\pi(s)\pi(s').$$

[1] Notice that if π and π' have common support, then the condition involving the supports is automatically satisfied.

Since $y \succeq_{\mathrm{Y}} y$, Assumption 2 implies that $y \succeq_{a_i} y$ for $i \in \{0,1\}$. Therefore, from the definition of $\succeq_{a_i}$ and the fact that π is TP_2, the above inequality is satisfied, and so $B(y,a,\pi)$ is indeed TP_2 for all $a \in \mathrm{A}$ and $y \in \mathrm{Y}$. By [1, Theorem 3], if either $B(y,a,\pi)$ or $B(y,a,\pi')$ is TP_2, then $B(y,a,\pi) \succeq_{\mathrm{tp}} B(y,a,\pi')$ if, and only if, $B(y,a,\pi) \succeq_r B(y,a,\pi')$. To finish, observe that if $s' \succeq_{\mathrm{S}} s$, then

$$B(y,a,\pi')(s')B(y,a,\pi)(s) \leq B(y,a,\pi')(s)B(y,a,\pi)(s') \Leftrightarrow \pi'(s')\pi(s) \leq \pi'(s)\pi(s'),$$

and the latter inequality holds, once more from the fact that π is TP_2.[2] □

References

[1] W. Whitt, Multivariate Monotone Likelihood Ratio and Uniform Conditional Stochastic Order, J. Appl. Probab., 19 (1982), 695-701.

[2]The analysis in [1] is restricted to probability measures on $\mathbb{R}^n$ that have densities either with respect to the Lebesgue measure or to some counting measure η. The case considered above can be reduced to the case in [1] by considering S as a subset of $\mathbb{R}^2$ and taking η to be such that $\eta(D) = |\mathrm{S} \cap D|/|\mathrm{S}|$ for all Borel subsets D of $\mathbb{R}^2$.